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A discontinuous wave-in-cell numerical scheme for hyperbolic conservation laws

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ABSTRACT

A new Riemann solver scheme for hyperbolic systems is introduced. The method consists of a discretization of the initial data into an approximate representation by discrete, discontinuous waves. Instead of calculating an intercell flux based on these waves, the discontinuous waves are propagated directly. Since the sum total of all discontinuous waves represents an extension of the linear Riemann problem, the solution is determined straightforwardly. For nonlinear systems, each timestep is considered a separate linear Riemann problem, and the projected waves are weighted to a background grid. This method is strikingly similar to the particle-in-cell approach, except that *discontinuous waves* are pushed around instead of macroparticles. The method is applied to Maxwell's equations and the equations of inviscid gasdynamics. For linear systems, exact solutions can be achieved without dissipation, and exact transmissive boundary condition treatments are trivial to implement. For inviscid gasdynamics, the nonlinear method tended to resolve discontinuities more sharply than the Roe, HLL or HLLC methods while requiring only between 30% and 50% of the execution time under identical conditions. The approach is also extremely robust, as it works for any Courant number. In the limit that the Courant number becomes infinite, the nonlinear solution approaches the linearized solution.

1. Introduction

1.1. Background

Since their inception in Godunov's seminal work [1], the Riemann solver class of methods have received growing attention and popularity for the discretization and numerical solution of nonlinear, hyperbolic partial differential equations [2–4]. These methods possess a robustness for dealing with discontinuities and steep gradients that may develop over time due to wave steepening, such as shock waves in the equations of fluid dynamics. Combined with the finite volume method, which is generally superior for preserving conserved quantities throughout the computational domain, these methods provide very useful shock-capturing approaches for nonlinear systems such as fluid dynamics.

These methods take advantage of the solution to the Riemann problem to calculate an upwinded intercell flux across two adjoining volumes. The methods can equivalently be seen as adding some numerical dissipation that stabilizes the average flux between the volumes. Godunov originally proposed solving the Riemann problem exactly, but this is computationally

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very expensive. Popular alternatives to the Godunov method include the HLL* (HLL, HLLC, HLLE, HLLM, etc.) Riemann solvers, the Roe Riemann solver, the Rusanov solver, and others.

The HLL [5] and HLLC [6] methods construct the Riemann solution using two-wave and three-wave models, respectively, for the equations of inviscid gasdynamics. These approaches require some estimation of the fastest signal speeds, the usual choices being those offered by Davis [7] and Einfeldt [8], and for the HLLC method, an additional wave estimate is needed provided by Toro [4]. Other choices of the signal speeds lead to variants of these approaches [4]. The Roe [9] method approximates the Riemann problem as a *linearized* problem with some constant, averaged Jacobian matrix; the exact solution to the linearized approach is then introduced. This method requires a full eigendecomposition available for the governing equations, which is not always possible [10]. The Rusanov [11] flux may be seen as a reduction of the Roe method in the case where a single maximum signal velocity is assumed instead of using the full eigenvalues of the governing equations.

Alternative finite volume methods that seek to calculate the flux without invoking the Riemann problem include the flux-vector splitting approaches. Popular choices include those pioneered by Sanders and Prendergast [12], Steger and Warming [13], van Leer [14], Liou and Steffan [15] and, more recently, Toro and Vazquez-Cendon [16]. While these methods enjoy simplicity of implementation and several desirable features, they tend to provide slightly more diffusive solutions than the Riemann solver class of approaches; on the other hand, they often require less eigendecomposition information and can be faster.

From a flux calculation perspective, the goal of any method is to calculate a reasonable numerical approximation to the amount of flux passing through adjoining volumes. The Godunov method takes advantage of the dependency of the flux function on the state; if a reasonable approximation of the state itself can be constructed, then the flux can be calculated directly from this state. The HLL* methods instead attempt to construct an approximation to the flux *directly*, without resorting to construction of the approximate state itself. The Roe method's assumption of a linearized problem allows direct calculation of the intercell flux from the constant average state.

Another entirely different approach to attempting to resolve the nonlinear behavior of the fluid dynamic equations uses the *particle-in-cell* (PIC) method [17]. PIC techniques for fluid dynamics introduce particles that advect fluid quantities in space, and then are weighted back to a grid [18]; this permits a form of solving the Lagrangian equations for fluid dynamics. The PIC approach has seen widespread success in simulations of plasma physics [17], where charged particles can be pushed around. At each timestep, the particles are weighted back to a grid, and the field equations are then solved for the next timestep.

A variant of the PIC method for fluid dynamics includes *smoothed particle hydrodynamics* (SPH), introduced by Lucy [19] and Gingold and Monaghan [20]. This method discards the use of a grid in favor of considering the particle to represent a smoothed volume, where its neighbors can experience molecular-like interactions to model complex physics. The method has seen widespread use, including areas such as magnetohydrodynamics [21]. In terms of resolving discontinuities, the method tends to be more diffuse than the Riemann solver or other state-of-the-art grid-based methods [22]. To circumvent this, both Riemann-solver-based [23–25] and non-Riemann-solver-based [22,26] modified SPH schemes have been proposed. The Riemann-solver-based SPH methods relate dissipation to those appearing in the intercell flux on the Riemann solver approaches [23], or calculate the force acting on each particle by solving the Riemann problem in the vicinity of the midpoint between interacting particles [24,25]. The non-Riemann-solver-based SPH schemes tend to use an adaptive density kernel estimation [22,26] instead.

1.2. Objectives

In this paper, our objective is to introduce a new Riemann solver approach that combines elements of both the classic grid-based approaches with the meshless approach of particle-in-cell and smoothed-particle hydrodynamics methods. The premier difference is the novel introduction of pushing *discrete, discontinuous waves* around within a computational domain, instead of discrete or smoothed particles. The method is essentially meshless; it is truly meshless for linear systems, but for the nonlinear case, the waves will be weighted back to a temporary background grid to reconstruct a new set of waves. Note that other approaches exist that may not require the use of a grid, even for nonlinear systems. The waves are birthed by considering the computational domain to represent a series of Riemann problems. In both cases, the waves are assumed to behave *linearly*; for the nonlinear case, the waves are weighted back and a new set of linear waves rebirthed from the grid to capture the nonlinear behavior. We christen the new method the *wave-in-cell* approach, since it treats the propagation of nonlinear waves in an analogous fashion as the particle-in-cell method treats the propagation of particles.

In context of the earlier discussion of how the flux vector is calculated for different Riemann solver approaches, this method avoids any calculation of the flux vector entirely. The new method essentially treats any general problem as a very large linear Riemann problem composed of all of the discontinuous waves in the entire domain. In the nonlinear case, this amounts to approximating *each timestep* as a single linear Riemann problem, composed of all of the waves in the entire domain; within each timestep, the wavespeeds are considered linear (corresponding to the known eigenvalues), and is updated at the end of each timestep. The solution to the linear Riemann problem may be calculated by summing over each discontinuous jump from an initial left or right state, and thus does not require the calculation of the flux vector at all.

In the *linear* case, a series of very rare and attractive properties regarding the solver will be presented. The method can actually provide exact solutions, in the case that the initial conditions consist only of discontinuities. It is *direct* in the sense that no time-marching or iteration is required; the solution may be evaluated immediately at any future time without