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High performance computations of steady-state bifurcations in 3D incompressible fluid flows by Asymptotic Numerical Method

Marc Medale [a,*], Bruno Cochelin [b]

[a] Aix-Marseille Université, CNRS, IUSTI UMR 7343, 5 rue Enrico Fermi, Technopole de Chateau-Gombert, 13453 Marseille Cedex 13, France
[b] Centrale Marseille, CNRS, LMA UPR 7051, Aix-Marseille Université, 13451 Marseille Cedex 20, France

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A B S T R A C T

This paper presents a powerful numerical model that implements the Asymptotic Numerical Method to compute 3D steady-state incompressible fluid flow solutions. This continuation algorithm enables to explore branches of steady-state solutions, stable or unstable, to accurately determine any simple steady-state bifurcation points and their emanating bifurcated branches. The powerfulness of the model stands on an optimal step length continuation thanks to the combination of power series analysis in the framework of ANM along with an efficient parallel implementation of the resulting algorithm on high performance computers. The outcome of this approach is demonstrated throughout 3D incompressible fluid flows inside a sudden expansion channel (expansion ratio $E = 3$, cross-section aspect ratio $10 \leq B \leq 20$). We have computed for the first time up to four steady symmetry breaking (pitchfork) bifurcations together with their associated bifurcated branches. The main characteristic of this 3D symmetric expansion configuration is that for a given cross-section aspect ratio the first bifurcation mode induces a top–bottom asymmetry, as in the 2D case, whereas the subsequent ones modulate the former in the span-wise direction with increasing wave numbers.

1. Introduction

To better understand, control and optimize complex physical phenomena or operating conditions in industrial processes, it is of first concern to know for which range of control parameters the system is either stable or unstable [25]. Indeed, at critical values of these control parameters, i.e. turning points, steady or Hopf bifurcation points, the stability of the system changes, so qualitative and quantitative changes in the system behavior may occur. Unfortunately, these critical points coincide with singular solution of the governing equations, so specialized algorithms should be used to accurately compute them [21,26,33]. In the computational fluid mechanics community, the classical way to determine a bifurcation diagram usually consists in performing the following sequence of two tasks for a series of control parameter values [13,20]: i) compute the base state associated with a given value of the control parameters; ii) compute the linear stability of this base state, as the system becomes unstable when the growth rate is greater than zero whereas it remains stable in the opposite case.

* Corresponding author.
E-mail address: marc.medale@univ-amu.fr (M. Medale).

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The present paper is only concerned with the first of these two steps, i.e., the computation of base states by continuation or path-following algorithms. These specialized methods are designed to map steady-state solutions in the parameter space by computing branches of solutions for a given range of control parameters, along with critical values and their corresponding singular solution, if any. Among continuation algorithms, first-order predictor–corrector ones (Euler predictor, Newton–Raphson based corrector) with pseudo-arc-length parameterization have been widely used for decades [26,22,32]. Nevertheless, it turns out that their step-length adaptivity may be in trouble in the vicinity of bifurcation points, resulting in weak computational efficiency and sometimes lack of convergence.

An alternate way to first-order predictor algorithms stands in high-order predictors that have been introduced in the Asymptotic Numerical Method (ANM) [17]. This method combines high-order Taylor series expansion, discretization technique and parameterization strategy, which results in a general and efficient non-linear solution method. It has been successfully applied in solid and structural mechanics problems, hydrodynamics ones [10,1,9,24], coupled fluid flow and heat transfer ones [27]. In the ANM step-length adaptivity is intimately related with the radius of convergence of the series, so it automatically shortens as the non-linearities increase and widens as the problem becomes softer. Replacing natural power series by rational ones (Padé approximates) enables to get step-lengths roughly twice as long as in the classical power series representation [16,23]. However, as continuation approaches to bifurcation points, whatever been the series representation step-length becomes very small and far from its optimal value.

The computation of bifurcation points and emanating branches are other particularly important issues. They have been tackled in the ANM framework by computing along the continuation either the zeros of Padé approximates or a scalar bifurcation indicator obtained from an additional perturbed problem solution [5,8,9,24]. Not only these methods have an extra cost which could be up to twice as the original ANM one, but also they could be sometimes lacking some robustness. Following Van Dyke's pioneer works on power series analysis [34–36], we have recently been able to not only accurately detect and compute simple bifurcation points in the course of continuation, but also build a new power series that recovers an optimal step-length in their vicinity, which significantly improves the overall computational efficiency of the ANM continuation algorithm [18].

The present paper introduces a high performance implementation of our continuation algorithm devoted to follow branches of steady-state solutions and locate steady bifurcation points in the framework of large size algebraic systems. They result from the discretization of 3D incompressible Navier–Stokes equations, for which the computation of bifurcation diagrams is still very challenging. The efficiency of our approach is highlighted by computing 3D bifurcation diagram of the incompressible fluid flow inside a sudden expansion channel. We have computed for the first time up to four steady symmetry breaking (pitchfork) bifurcations together with their associated bifurcated branches. It turns out that the first bifurcation mode induces a top–bottom asymmetry, as in the 2D case, whereas the subsequent ones modulate the former in the span-wise direction with increasing wave numbers.

The paper is organized as follows. The ANM continuation algorithm associated with power series analysis is presented in Section 2 for both regular and singular starting points. Then, the key points of its implementation for 3D incompressible Navier–Stokes are discussed in Section 3 in the framework of high performance computing. Section 4 presents an outcome example of this ANM implementation throughout the study 3D fluid flows inside a channel with symmetric sudden expansion. Finally, concluding remarks and promising directions are discussed in Section 5.

2. Continuation algorithm based on the ANM

Let us first recall some minimal background related to the ANM framework. Let $R(u,\lambda)=0$ be an algebraic system of n non-linear smooth equations, where $u\in\mathbb{R}^n$ is a vector of state variables and $\lambda\in\mathbb{R}$ a single control parameter. The extended state vector $U=\begin{bmatrix}u\\ \lambda\end{bmatrix}\in\mathbb{R}^{n+1}$ is introduced for compactness as it includes the parameter λ as its last state variable, so that the equilibrium system now reads:

$$R(U)=L0+L(U)+Q(U,U)=0 \tag{1}$$

in which one assumes for the sake of simplicity that the vector equation is polynomial and quadratic in U and made up of $L0$ a constant vector, $L(.)$ a linear operator and $Q(.,*)$ a bilinear operator. This assumption is not mandatory but, on the one hand it nicely fits the Navier–Stokes equations implemented in this paper and on the other hand it is very convenient to ease the derivation of Taylor series at the heart to the ANM algorithm.

Generic solutions of eq. (1) are branches of solutions, which are represented in ANM continuation algorithms by power series expansion of eq. (2):

$$U(a)=U_0+a\,U_1+a^2\,U_2+\cdots+a^m\,U_m \tag{2}$$

where m is the truncate order of the power series expansion and a the path parameter defined in the parameterization equation (3) as the classical pseudo arc-length [14,17]:

$$a=[U(a)-U_0]^T\cdot U_1 \tag{3}$$

Obviously, others choices could equivalently be done among parameterization techniques compatible with the ANM, such as local parameterization or residual minimization [28].