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Divergence-free MHD on unstructured meshes using high order finite volume schemes based on multidimensional Riemann solvers

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ABSTRACT

Several advances have been reported in the recent literature on divergence-free finite volume schemes for Magnetohydrodynamics (MHD). Almost all of these advances are restricted to structured meshes. To retain full geometric versatility, however, it is also very important to make analogous advances in divergence-free schemes for MHD on unstructured meshes. Such schemes utilize a staggered Yee-type mesh, where all hydrodynamic quantities (mass, momentum and energy density) are cell-centered, while the magnetic fields are face-centered and the electric fields, which are so useful for the time update of the magnetic field, are centered at the edges.

Three important advances are brought together in this paper in order to make it possible to have high order accurate finite volume schemes for the MHD equations on unstructured meshes. First, it is shown that a divergence-free WENO reconstruction of the magnetic field can be developed for unstructured meshes in two and three space dimensions using a classical cell-centered WENO algorithm, without the need to do a WENO reconstruction for the magnetic field on the faces. This is achieved via a novel constrained L_2-projection operator that is used in each time step as a postprocessor of the cell-centered WENO reconstruction so that the magnetic field becomes locally and globally divergence free. Second, it is shown that recently-developed genuinely multidimensional Riemann solvers (called MuSIC Riemann solvers) can be used on unstructured meshes to obtain a multidimensionally upwinded representation of the electric field at each edge. Third, the above two innovations work well together with a high order accurate one-step ADER time stepping strategy, which requires the divergence-free nonlinear WENO reconstruction procedure to be carried out only once per time step.

The resulting divergence-free ADER–WENO schemes with MuSIC Riemann solvers give us an efficient and easily-implemented strategy for divergence-free MHD on unstructured meshes. Several stringent two- and three-dimensional problems are shown to work well with the methods presented here.

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1. Introduction

The magnetohydrodynamic (MHD) equations have become increasingly important in astrophysics, space physics and plasma physics. MHD is also the simplest approximation in a hierarchy of approximations for modelling ionized plasmas. Many novel computational insights, which are inapplicable to computational fluid dynamics, have to be developed for the numerical solution of the compressible MHD equations. Those insights can also be used in other more sophisticated approximations for modelling ionized plasmas. For MHD, the ionized plasma couples to the magnetic field which evolves according to Faraday's law:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{1.1}$$

In the ideal MHD approximation, we have $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, where $\mathbf{B}$ is the magnetic field, $\mathbf{E}$ is the electric field and $\mathbf{v}$ is the fluid velocity. Since the magnetic field must be divergence-free at the initial time, i.e. $\nabla \cdot \mathbf{B} = 0$, Faraday's law ensures that the magnetic field remains divergence-free for all time. Physically, this means that no magnetic monopoles can exist. Good numerical methods for MHD ought to retain this property. In a very influential paper, Brackbill and Barnes [22] showed that if a non-zero divergence of $\mathbf{B}$ is allowed to build up in the computational domain, it can lead to unphysical plasma transport parallel to the magnetic field.

To promote fidelity with the physics, several schemes have been devised that keep the magnetic field divergence-free. Fundamentally, all these methods use a staggered Yee-type [74] mesh where the normal components of the magnetic fields reside in the faces of the mesh and the electric fields reside at the edges of the mesh. Such schemes are also known as constrained transport schemes because they transport the magnetic field consistent with the divergence-free constraint. Early constrained transport schemes were devised without the use of higher order Godunov methods (Brecht et al. [23], DeVore [33], Evans and Hawley [41]).

Over the last decade or two, several effective one-dimensional Riemann solvers have become available for numerical MHD (Brio and Wu [24], Zachary et al. [75], Dai and Woodward [30], Ryu and Jones [62], Roe and Balsara [61], Cargo and Gallice [25], Balsara [3], Falle et al. [42], Gurski [45], Li [53], Miyoshi and Kusano [56]). This has resulted in several Godunov schemes for numerical MHD. Many of the earlier such schemes did not respect the divergence-free constraint (Brackbill and Barnes [22], Zachary et al. [75], Crockett et al. [29], Balsara [4], Balbas et al. [2]). Most such schemes resort to some strategy for reducing the unbounded growth of divergence in the magnetic field. Hodge projection approaches have been suggested (Zachary et al. [75], Balsara [4]) and the deficiencies of the Hodge projection have been catalogued by Balsara and Kim [10]. The Powell [59] source term formulation advects away any divergence that might build up, but it only does so by introducing source terms that destroy the conservation properties of the momentum and energy equations. The generalized Lagrange multiplier (GLM) approach by Dedner et al. [32] does provide conservation of momentum and energy, however Mocz et al. [57] present some problems where the method proves deficient. The GLM method has the further problem that the speed with which the GLM field needs to propagate has to be faster than the fastest speed in the problem. In some simulations, this top speed can grow by orders of magnitude as the simulation evolves, making it difficult to predict the top speed. Modern higher order Godunov schemes for numerical MHD tend to incorporate the divergence-free property for the magnetic field (Dai and Woodward [31], Ryu et al. [63], Balsara and Spicer [6], Toth [69], Londrillo and DelZanna [54], Gardiner and Stone [43,44], Balsara [7–9], Lee [50]). Balsara and Spicer [6] suggested that the dualism between the components of the numerical flux and the electric field can be used to obtain the electric field at the edges of each zone. All higher order, divergence-free Godunov schemes for numerical MHD have incorporated this plan in one form or another.

A reading of the later sections of Balsara and Spicer [6] shows that the electric field has to be obtained in a multidimensional manner. Let us focus on the equations of MHD in flux form to understand the issue.

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \varepsilon \\ B_x \\ B_y \\ B_z \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho v_x \\ \rho v_x v_x + P + \mathbf{B}^2/8\pi - B_x B_x/4\pi \\ \rho v_y v_x - B_x B_y/4\pi \\ \rho v_z v_x - B_x B_z/4\pi \\ (\varepsilon + P + \mathbf{B}^2/8\pi) v_x - B_x(\mathbf{v}\cdot\mathbf{B})/4\pi \\ 0 \\ (v_x B_y - v_y B_x) \\ -(v_z B_x - v_x B_z) \end{pmatrix}$$