Journal of Computational Physics 298 (2015) 305–319

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Immersed finite elements for optimal control problems of elliptic PDEs with interfaces

Qian Zhang [a], Kazufumi Ito [b], Zhilin Li [b,a], Zhiyue Zhang [a,*]

[a] School of Mathematical Sciences, Jiangsu Key Laboratory for NSLSCS, Nanjing Normal University, Nanjing 210023, China
[b] Department of Mathematics, North Carolina State University, 27695, USA

ARTICLE INFO

Article history:
Received 1 January 2015
Received in revised form 9 May 2015
Accepted 28 May 2015
Available online 17 June 2015

Keywords:
PDE-constrained optimization
Variational discretization
Immersed finite element
Elliptic interface problem

ABSTRACT

This paper presents a numerical method and analysis, based on the variational discretization concept, for optimal control problems governed by elliptic PDEs with interfaces. The method uses a simple uniform mesh which is independent of the interface. Due to the jump of the coefficient across the interface, the standard linear finite element method cannot achieve optimal convergence when the uniform mesh is used. Therefore the immersed finite element method (IFEM) developed in Li et al. [20] is used to discretize the state equation required in the variational discretization approach. Optimal error estimates for the control, state and adjoint state are derived. Numerical examples are provided to confirm the theoretical results.

1. Introduction

Optimal control problems governed by elliptic PDEs with interfaces arise in many applications, such as the optimization or optimal control of a process in a domain which is composed of several materials separated by curves or surfaces (called interfaces). Coefficients in the elliptic PDEs may have a jump across the interface corresponding to different materials. Hence, it is a challenge to develop efficient numerical methods for such optimal control problems.

Elliptic interface problems have been extensively discussed in the literature. We use a uniform Cartesian mesh in our method. One consideration is that it is difficult and time consuming to generate body fitted meshes for complicated or moving interfaces. How to design accurate methods on unfitted meshes has attracted a lot of attention in the literature. The immersed finite element method (IFEM) proposed in [20] is among a few methods that based on linear finite element discretizations and unfitted meshes, for example, uniform triangulations. The idea of the IFEM is to modify the basis functions in the interface triangles so that the interface conditions are satisfied. Optimal approximation capabilities of the immersed finite element space have been proved in [19]. And optimal error estimates in L^2 and H^1 norms have been given in [10].

Numerical methods for optimal control problems governed by elliptic PDEs have been discussed in many publications (see, e.g., [1,5–8,11–14,17,18,21,22]). However, to the best of our knowledge, there are few papers that concern the numerical method based on unfitted meshes for the optimal control problems governed by elliptic PDEs with interfaces. In [2], Apel and Sirch considered the distributed optimal control problem governed by elliptic equation with discontinuous coefficients. The diffusion coefficient has different values on polygonal Lipschitz subdomains. To avoid a reduced convergence order, graded meshes are used.

* Corresponding author.
E-mail address: zhangzhiyue@njnu.edu.cn (Z. Zhang).

http://dx.doi.org/10.1016/j.jcp.2015.05.050
0021-9991/

In this paper, we try to discretize the distributed control problem governed by elliptic interface problems by combining the variational discretization concept and the IFEM on uniform triangulations. Optimal error estimates are derived for the control, state and adjoint state, which are the same as that obtained by standard finite element methods for the control problem without interfaces. In the case of the control without constraints, the discretization leads to a symmetric but indefinite system of equations. A block diagonally preconditioned MINRES algorithm [23,24] is used to solve the indefinite system. In the case of the control with constraints, a nonlinear and non-smooth equation for the discrete control is obtained. A numerically implementable fix-point iteration is used to solve that single equation for the discrete control. Finally, some numerical examples with or without control constraints are provided to verify the theoretical analysis. We also compare the numerical results obtained by the IFEM and the standard linear FEM on the same mesh respectively. One can see that the numerical method based on the IFEM improves the accuracy significantly.

The remainder of this paper is organized as follows. In Section 2, the model problem is introduced and optimality conditions and regularity results of the problem are given. Section 3 presents the discretization of the optimal control problem based on the variational discretization approach and the immersed finite element method. And some error estimates are derived. In Section 4, the implementation details of the numerical method are described briefly. In Section 5, numerical examples are provided to confirm the theoretical results. Some conclusions are made in Section 6.

2. Model problem and optimality conditions

Consider the elliptic interface problem,

$$-\nabla\cdot(\beta(\mathbf{x})\nabla y(\mathbf{x})) = u(\mathbf{x}) \quad \text{in } \Omega\backslash\Gamma, \tag{2.1}$$

$$[y]_\Gamma = 0, \qquad [\beta\partial_{\mathbf{n}} y]_\Gamma = 0, \tag{2.2}$$

$$y = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where $\Omega \in \mathbb{R}^2$ is a bounded domain separated by a closed interface $\Gamma \in C^2$, $[v]_\Gamma$ denotes the jump of the function $v(\mathbf{x})$ across an interface Γ. We assume that the interface Γ separates the domain Ω into two sub-domains Ω^+ and Ω^-, and Ω^- lies strictly inside Ω, see Fig. 2 for an illustration. The vector $\mathbf{n}$ is the unit normal direction of Γ pointing to Ω^+. The coefficient $\beta(\mathbf{x})$ is a positive and piecewise constant, that is,

$$\beta(\mathbf{x}) = \beta^+ \text{ if } \mathbf{x}\in\Omega^+, \qquad \beta(\mathbf{x}) = \beta^- \text{ if } \mathbf{x}\in\Omega^-. \tag{2.4}$$

The weak formulation of the state equations (2.1)–(2.3) is to

$$\text{find } y \in H_0^1(\Omega) \text{ such that } a(y,v) = (u,v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega), \tag{2.5}$$

where $a(y,v) = \sum_{s=\pm}\int_{\Omega^s}\beta^s\nabla y\cdot\nabla v d\mathbf{x}$ and $(u,v)_{L^2(\Omega)} = \int_\Omega uv d\mathbf{x}$.

Problem 2.1. (**P**) Consider the optimal control problem of minimizing

$$J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2 d\mathbf{x} + \frac{\alpha}{2}\int_\Omega u^2 d\mathbf{x} \tag{2.6}$$

over all $(y,u) \in H_0^1(\Omega)\times L^2(\Omega)$ subject to the elliptic interface problem (2.1)–(2.3) and the control constraints

$$u_a \le u \le u_b. \tag{2.7}$$

The regularization parameter α is a fixed positive number and the set of admissible controls for (**P**) can be written as

$$U_{ad} = \left\{u \in L^2(\Omega) : u_a \le u \le u_b\right\}.$$

We make the following smoothness assumption on the data of the problem, that is, $y_d \in L^2(\Omega)$, and $u_a,\ u_b \in L^2(\Omega)$.

Since the problem is quadratic and convex, by applying standard techniques see [16,25], we have the existence of solutions and the optimality conditions.

Theorem 2.2. *The problem* (**P**) *admits a unique optimal control $u^* \in L^2(\Omega)$, with an associated state $y^* \in H_0^1(\Omega)$ and an adjoint state $p^* \in H_0^1(\Omega)$ that satisfy the state equation*

$$a(y^*,v) = (u^*,v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega), \tag{2.8}$$

the adjoint equation

$$a(v,p^*) = (y^*-y_d,v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega), \tag{2.9}$$