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Some high-order difference schemes for the distributed-order differential equations

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ABSTRACT

Two difference schemes are derived for both one-dimensional and two-dimensional distributed-order differential equations. It is proved that the schemes are unconditionally stable and convergent in an $L^1(L^\infty)$ norm with the convergence orders $O(\tau^2+h^2+\Delta\alpha^2)$ and $O(\tau^2+h^4+\Delta\alpha^4)$, respectively, where τ, h and $\Delta\alpha$ are the step sizes in time, space and distributed-order variables. Several numerical examples are given to confirm the theoretical results.

1. Introduction

In recent years, the research on fractional calculus has gained a growth interest due to its powerful potential to depict many processes in physics, engineering, finance, material science, control system, signal processing and so on. The related fractional differential equation has been studied from different viewpoints. For most of them, analytical solutions are not available or too complicated to compute, so some effective numerical methods are resorted to and there have been quite abundant literatures on this subject, in which the finite difference method is one popular way.

For the single-order time-fractional diffusion equation, the finite difference method has been well studied in recent few years. We now here only mention a few part of related publications, such as the work by Yuste and Acedo [1], Langlands and Henry [2], Chen et al. [3], Zhuang et al. [4], Cao and Xu [5], Li and Ding [6], Sun and Wu [7]. For more results, readers can refer to the review article [8] and the references therein.

It is pointed out recently that the time-fractional anomalous diffusion equation with a constant-order temporal derivative cannot describe the processes lacking temporal scaling, so in some sense, the time-fractional diffusion equation of distributed order was introduced to describe processes getting more anomalous in course of time, e.g. the accelerating or retarded sub-diffusion [9]. The distributed-order differential equation can also be regarded as a natural generalization of the single-order and the multi-term fractional differential equation [10]. The idea of distributed-order differential equation was first proposed probably by Caputo in [11], which was also stated by Podlubny et al. in [12]. Jiao, Chen and Podlubny [13] presented a concise and insightful view to understand the usefulness of distributed-order concept in control and sig-

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nal processing. Luchko [14] investigated some uniqueness and existence results of solutions to boundary value problems of the generalized distributed-order time-fractional diffusion equation by an appropriate maximum principle. Gorenflo et al. [15] obtained a representation of the fundamental solution to the Cauchy problem of a distributed-order time-fractional diffusion-wave equation by employing the technique of the Fourier and Laplace transforms and gave the interpretation of the fundamental solution as a probability density function. For more general distributed-order differential equations, analytical solutions are not easy to obtain. Therefore, the consideration on numerical methods to solve the distributed-order differential equation should be involved.

Generally speaking, when numerically solving the distributed-order differential equations, the approximation to the distributed integral is taken into account one step ahead, using the classical numerical quadrature formulae. Then the original distributed-order problem is approximated by a multi-term fractional differential problem. What follows is how to efficiently solve the approximated multi-term fractional differential problem. So far, to our knowledge, only a few works have been concerned with this issue. Podlubny et al. [12] proposed a matrix approach method to solve the distributed-order ordinary differential equation. Diethelm and Ford [10] provided some numerical analysis for distributed-order differential equations, where the equivalent system of single-order differential equations was used to treat the multi-term time-fractional differential problem after the numerical approximation by the quadrature formula for the distributed-order integral. Then a fractional linear multi-step method or an Adams-type predictor–corrector scheme was brought to solve the equivalent system. The similar techniques to deal with the multi-term fractional differential equation are discussed in recent work [16]. Recently, Ye et al. [17] constructed an implicit difference scheme for the time distributed-order and Riesz space fractional diffusions on bounded domains. The unconditional stability and convergence were proved by mathematical induction method. Ford, Morgado and Rebelo [18] discussed a numerical method for the distributed-order time-fractional diffusion equation. Morgado and Rebelo [19] presented an implicit scheme for the numerical approximation of the distributed order time-fractional reaction–diffusion equation with a nonlinear source term. The midpoint rule was used to approximate the distributed integral in [17–19], and the same discrete $L1$ formula was used to approximate the involved Caputo fractional derivatives, thus the numerical accuracies of the resultant schemes in time variable were all about first order. Katsikadelis [20] presented a numerical method for distributed-order differential equations, where the trapezoidal rule was used to discretize the distributed integral and the analog equation method was employed to solve the approximated multi-term fractional differential equation. The stability and convergence were only shown by numerical examples without any strict proof. To our knowledge, the high-order schemes to numerically solve the distributed-order differential equations have not been seen.

In addition, the bottleneck of numerical methods for solving the time-fractional differential problem lies in the global storage and computation of unknowns at all previous time layers when the problem is approximated at the considered time layer, which is caused by the nonlocal property of fractional operators. So the high order numerical methods have been pursued for alleviating this drawback. Spatially compact techniques have been widely used to enhance the space numerical accuracy when solving the time-fractional differential problem. Major results along this routine cover the work by Cui [21,22], Chen et al. [23], Hu and Zhang [24,25], Sun et al. [26–29] and so on.

Our work here will apply the weighted and shifted Grünwald formula, which was proposed in [30] and used in [31], to approximate the involved time-fractional derivatives. The numerical accuracy in time can achieve the global second order independent of orders of fractional derivatives. On the other hand, we will also make our attempt to explore the numerical solutions with high-order accuracy in space for solving the distributed-order differential equation.

Consider the following described distributed-order problem

$$\mathcal{D}_t^w u(\mathbf{x},t) = \Delta u(\mathbf{x},t) + F(\mathbf{x},t), \quad \mathbf{x}\in\Omega, \quad 0<t\leqslant T, \tag{1.1}$$

$$u(\mathbf{x},0) = 0, \quad \mathbf{x}\in\overline{\Omega}, \tag{1.2}$$

$$u(\mathbf{x},t)|_{\mathbf{x}\in\partial\Omega} = \psi(\mathbf{x},t), \quad 0<t\leqslant T, \tag{1.3}$$

where Δ is the Laplacian operator, $\partial\Omega$ is the boundary of Ω, $F(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ are given functions,

$$\psi(\mathbf{x},0) = 0, \ \text{if } \mathbf{x}\in\partial\Omega, \qquad \mathcal{D}_t^w u(\mathbf{x},t) = \int_0^1 w(\alpha)\, {}_0^C D_t^\alpha u(\mathbf{x},t)\mathrm{d}\alpha,$$

$$w(\alpha)\geqslant 0, \quad \int_0^1 w(\alpha)\mathrm{d}\alpha = c_0 > 0,$$

$${}_0^C D_t^\alpha u(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{-\alpha}\frac{\partial u}{\partial\xi}(\mathbf{x},\xi)\mathrm{d}\xi \ \text{for } 0\leqslant\alpha<1, \ {}_0^C D_t^\alpha u(\mathbf{x},t) = u_t(\mathbf{x},t) \ \text{for } \alpha = 1.$$

It is noted that the zero initial value case is only considered here. If it is not so, then an auxiliary unknown function can be introduced to return it to the zero initial value problem.