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# Wave propagation in anisotropic elastic materials and curvilinear coordinates using a summation-by-parts finite difference method

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ABSTRACT

We develop a fourth order accurate finite difference method for solving the three-dimensional elastic wave equation in general heterogeneous anisotropic materials on curvilinear grids. The proposed method is an extension of the method for isotropic materials, previously described in the paper by Sjögreen and Petersson (2012) [11]. The proposed method discretizes the anisotropic elastic wave equation in second order formulation, using a node centered finite difference method that satisfies the principle of summation by parts. The summation by parts technique results in a provably stable numerical method that is energy conserving. We also generalize and evaluate the super-grid far-field technique for truncating unbounded domains. Unlike the commonly used perfectly matched layers (PML), the super-grid technique is stable for general anisotropic material, because it is based on a coordinate stretching combined with an artificial dissipation. As a result, the discretization satisfies an energy estimate, proving that the numerical approximation is stable. We demonstrate by numerical experiments that sufficiently wide super-grid layers result in very small artificial reflections. Applications of the proposed method are demonstrated by three-dimensional simulations of anisotropic wave propagation in crystals.

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## 1. Introduction

This paper describes a fourth order accurate numerical method for calculating wave propagation in general anisotropic elastic materials, i.e., materials in which waves propagate with different speeds in different directions. Such materials occur in several applications. One class of anisotropic materials are crystals. Here the directionally dependent wave propagation properties follow from the symmetries and structure of the atomic bonds in the crystal. In seismic applications, isotropic layered materials behave anisotropically when they are subjected to waves where the wavelength is much longer than the thickness of the layers [1]. Furthermore, fractures in an isotropic material can lead to directionally dependent wave propagation properties [2], i.e., anisotropic behavior. More generally, spatial homogenization of a fine grained heterogeneous isotropic elastic material is known to result in a coarser grained elastic model with anisotropic properties [3,4].

Many wave propagation codes for isotropic elastic materials are based on finite difference methods on staggered grids [5–7]. These methods approximate the elastic wave equation in first order velocity-stress formulation. Unfortunately,

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the staggered grid approach is non-trivial to generalize to general anisotropic materials. The fundamental difficulty is to place the dependent variables on the staggered grid, such that all terms in the anisotropic Hooke's law can be approximated accurately and, at the same time, making the numerical method stable. Since an isotropic material has anisotropic properties when the equations are transformed to curvilinear coordinates, similar difficulties occur for staggered grid methods on curvilinear meshes. Node centered methods, which discretize the elastic wave equation in second order displacement formulation, do not have this difficulty. For example, the spectral element method, described in [8], is naturally formulated for general linear stress-strain relationships, and has successfully been used for modeling general anisotropy [9] as well as realistic topography using curvilinear (unstructured hexahedral) meshes [10].

The present paper has two objectives. First, we describe a fourth order accurate node centered finite difference scheme for wave propagation in general anisotropic elastic materials. Our scheme satisfies the principle of summation by parts (SBP) and is a generalization of the method described in [11,12], which is implemented in the elastic wave propagation open source code SW4, version 1.0 [13]. The finite difference scheme is fourth order accurate, stable, and energy conserving. We here generalize the method to a fully anisotropic material in curvilinear coordinates, allowing for accurate modeling of realistic topography. Our main motivation for using the summation by parts method is to obtain a spatial discretization that satisfies an energy estimate, which guarantees stability of the numerical approximation for heterogeneous materials on curvilinear grids with free surface or Dirichlet boundary conditions. We remark that our SBP method uses ghost points just outside the boundaries to enforce the boundary conditions in a strong (point-wise) sense. There is a related SBP method that does not use ghost points and instead enforces the boundary conditions in a weak sense using penalty techniques, see e.g. [14–16].

The second objective is to analyze and numerically evaluate the super-grid far-field truncation technique for anisotropic elastic materials. Super-grid far-field conditions truncate very large or unbounded domains to finite extent by adding sponge layers outside the domain of interest. Inside the layers, the wave equation is modified by a combination of grid stretching and high order artificial dissipation. Compared to perfectly matched layers (PML) [17], the greatest strength of the super-grid technique is that the overall numerical method is provably stable, as long as the underlying numerical method is stable on a curvilinear grid. Note that the PML equations can have unstable solutions (growing exponentially in time) for some anisotropic materials [18] that violate the so-called geometric stability condition [19]. We have previously proven that the *isotropic* discretized elastic wave equation with super-grid layers satisfies an energy estimate [12], precluding exponential growth of the numerical solution. In the present paper, we extend that analysis to general *anisotropic* elastic materials on curvilinear grids. An additional strength of the super-grid technique is its simplicity and low computational cost. In contrast to the PML method, super-grid does *not* rely on augmenting the wave equation with additional differential equations that govern additional dependent variables. A potential weakness of the super-grid technique is that it does not achieve the 'perfect' non-reflecting property of PML. However, numerical experiments indicate that artificial reflections decay rapidly with the width of the super-grid layers.

This paper is organized as follows. In Section 2, we review the equations of anisotropic elastic wave propagation in Cartesian coordinates. Section 3 generalizes the results of Section 2 to curvilinear coordinates. The finite difference discretization is presented in Section 4, where we also present an efficient way of estimating the stability limit for the time step. Section 5 describes the super-grid technique and numerical experiments are presented in Section 6. Here we verify the accuracy of the proposed finite difference scheme, demonstrate the accuracy of the super-grid far-field truncation technique, and verify energy conservation of the numerical solution. Conclusions are given in Section 7.

## 2. The anisotropic elastic wave equation

To make the presentation self-contained and to define a consistent notation, we start by introducing the governing equations in a form that is amenable for constructing the SBP discretization. For further background information, we recommend an advanced text book on solid mechanics, e.g. [20].

We consider the time-dependent elastic wave equation in a three-dimensional domain $\mathbf{x} \in \Omega$, where $\mathbf{x} = (x^{(1)}, x^{(2)}, x^{(3)})^T$ are the Cartesian coordinates and $\mathbf{u} = (u^{(1)}, u^{(2)}, u^{(3)})^T$ are the Cartesian components of the three-dimensional displacement vector. In displacement formulation, the elastic wave equation takes the form

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \mathcal{T} + \mathbf{F}, \quad \mathbf{x} \in \Omega, \quad t \geq 0, \tag{1}$$

$$\nabla \cdot \mathcal{T} = G_s^T C G_s \mathbf{u} =: \mathbf{L}\mathbf{u}, \tag{2}$$

subject to appropriate initial and boundary conditions. Here, $\rho$ is the density, $\mathcal{T}$ is the stress tensor, and $\mathbf{F}$ is the external forcing per unit volume. The spatial operator $\mathbf{L}$ is called the $3 \times 3$ symmetric Kelvin–Christoffel differential operator matrix [20]. Let $\mathcal{T}_{ij}$ and $\mathcal{E}_{ij}$ be the Cartesian components of the symmetric stress and strain tensors, respectively. We adopt Voigt's vector notation,

$$\boldsymbol{\sigma} = (\mathcal{T}_{11}, \mathcal{T}_{22}, \mathcal{T}_{33}, \mathcal{T}_{23}, \mathcal{T}_{13}, \mathcal{T}_{12})^T, \quad \mathbf{e} = (\mathcal{E}_{11}, \mathcal{E}_{22}, \mathcal{E}_{33}, 2\mathcal{E}_{23}, 2\mathcal{E}_{13}, 2\mathcal{E}_{12})^T,$$

which allows Hooke's law to be expressed in terms of the $6 \times 6$ stiffness matrix $C$, which is symmetric and positive definite [20]. Because $C$ is symmetric, it has 21 unique elements, corresponding to the 21 parameters of a general anisotropic