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A preconditioned fast finite volume scheme for a fractional differential equation discretized on a locally refined composite mesh

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ABSTRACT

Numerical methods for fractional differential equations generate full stiffness matrices, which were traditionally solved via Gaussian type direct solvers that require $O(N^3)$ of computational work and $O(N^2)$ of memory to store where N is the number of spatial grid points in the discretization. We develop a preconditioned fast Krylov subspace iterative method for the efficient and faithful solution of finite volume schemes defined on a locally refined composite mesh for fractional differential equations to resolve boundary layers of the solutions. Numerical results are presented to show the utility of the method.

1. Introduction

Fractional differential equations were shown to provide an adequate and accurate description of transport processes that exhibit anomalous diffusive behavior, which cannot be modeled properly by canonical second-order diffusion equations [2,15]. Extensive research has been conducted in the development of numerical methods for fractional differential equations [5,7,9,10,12–14,18,26–28]. Because of the nonlocal nature of fractional differential operators, numerical methods for space-fractional differential equations usually generate full stiffness matrices and were traditionally solved via Gaussian elimination. This requires $O(N^3)$ of computational work per time step and $O(N^2)$ of memory to store, where N is the number of spatial grid points in the discretization, and represents a significant increase over the computational cost and memory requirement of numerical methods of second-order diffusion equations.

We proved that the stiffness matrices of finite difference schemes for space-fractional differential equations can be decomposed as a sum of diagonal-multiply-Toeplitz matrices (or their block analogue in multidimensional cases), and consequently developed fast solution methods that has a computational work account of $O(N \log N)$ per iteration or time step and has a memory requirement of $O(N)$ via the fast Fourier transform (FFT), while retaining the accuracy of the underlying numerical schemes [19,21,23]. A fast finite volume scheme was also developed in [20]. Numerical experiments showed the significant reduction in computational cost and memory requirement over the traditional methods. As they were based upon FFT, these fast methods were limited to uniform spatial meshes.

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It was not realized until recently [11,24,25] that solutions to fractional differential equations may exhibit boundary layer and poor regularity even if the diffusivity coefficient and right-hand side are smooth. For example, it is easy to check that

$$u(x) = x^{1-\beta}, \quad x \in (0,1), \tag{1}$$

with $0 < \beta < 1$, is the solution to the boundary-value problem of the homogeneous fractional differential equation with a constant diffusivity coefficient

$$\begin{aligned} &D\big({}_0D_x^{-\beta} Du\big) = 0, \quad x \in (0,1), \\ &u(0) = 0, \quad u(1) = 1 \end{aligned} \tag{2}$$

where D represents the first-order differential operator, and ${}_0D_x^{-\beta}$ and ${}_xD_1^{-\beta}$ represent the left and right fractional integral operators [16]

$$\begin{cases} {}_0D_x^{-\beta} g(x) := \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^x (x-s)^{\beta-1} g(s)ds, \\ {}_xD_1^{-\beta} g(x) := \dfrac{1}{\Gamma(\beta)} \displaystyle\int_x^1 (s-x)^{\beta-1} g(s)ds \end{cases} \tag{3}$$

where $\Gamma(\cdot)$ is a Gamma function.

It is clear that the solution u has a boundary layer. It was proved in [25] that the true solution u is not in the fractional Sobolev space $H^{\frac{3}{2}-\beta}(0,1)$, but is still in the Besov space $B_\infty^{\frac{3}{2}-\beta}(L^2(0,1))$ [1]. From a numerical point of view, a numerical scheme with a uniform mesh is probably not anticipated to efficiently resolve the boundary layer of the solution u at $x = 0$, and so a numerical scheme that is discretized on a locally refined composite mesh is desired. This, in turn, implies that fractional finite difference methods are out of the question, as they are based on the discretization of the Grünwald–Letnikov fractional derivatives that are inherently defined on uniform meshes [16].

A fundamental approach to resolve fronts in solutions to differential equations is to employ an adaptive mechanism. However, in the context of fractional differential equations, any local change in a mesh will destroy the global structure of the stiffness matrix. Hence, one should take great care in the development of an adaptive method by delicately balancing the adaptivity of the method and the structure of the adapted meshes. As a preliminary first step towards the eventual resolution of the steep fronts present in the solutions to fractional differential equations, the goal of this paper is to derive a fast and faithful finite volume scheme on a locally refined composite mesh for the inhomogeneous Dirichlet boundary-value problem of the conservative variable-coefficient Caputo fractional differential equation of order $2-\beta$ with $0 < \beta < 1$ [2,6,7]

$$\begin{aligned} &-D\big(K(x)\big(\gamma\, {}_0D_x^{-\beta} + (1-\gamma)\, {}_xD_1^{-\beta}\big)Du\big) = f(x), \quad 0 < x < 1, \\ &u(0) = u_l, \qquad u(1) = u_r. \end{aligned} \tag{4}$$

Here $K(x)$ is the diffusivity coefficient, $0 \le \gamma \le 1$ indicates the relative weight of forward versus backward transition probability, $f(x)$ is the source and sink term, and u_l and u_r are the prescribed Dirichlet boundary data. The rest of the paper is organized as follows. In Section 2 we derive a finite volume scheme on a composite mesh. In Section 3 we study the structure of the stiffness matrix. In Section 4 we develop a fast and faithful Krylov subspace method for the finite volume scheme. In Section 5 we present an efficient preconditioner. In Section 6 we carry out numerical experiments to investigate the performance of the fast method. In Section 7 we discuss extensions and future directions.

2. A finite volume scheme on a composite mesh

For simplicity of presentation, we assume that the boundary layer of the true solution is located at the left endpoint $x = 0$. We will consider the general case that the boundary layers of the true solution appear at both endpoints $x = 0$ and $x = 1$ at the end of the paper. To resolve the potential boundary layer of the true solution while maximizing the efficiency of the derived numerical scheme, we introduce a composite mesh as follows:

We begin by a uniform partition of mesh size $h := 1/n$ for a positive integer n. Then we introduce a geometrically decreased mesh on the subinterval $[0, h]$ starting from $x = h$ successfully for m times with m being a positive number. Let $N := m + n$. We follow the convention to label the nodes in the composite mesh from left to right as follows: we let $h_1 := 2^{-m}h$ be the finest mesh, $h_i := 2^{-(m-(i-2))}h$ for $i = 2, \cdots, m+1$ be the mesh sizes of a geometrically increasing mesh, and $h_i := h$ for $i = m+2, \cdots, N$ be the uniformly coarse mesh size where. Then we set $x_0 := 0$ and $x_i := x_{i-1} + h_i$ for $i = 1, 2, \cdots, N$ to be the sequence of space nodes of the composite mesh. It is clear that $x_{m+1} = h$ and $x_N = 1$.

Let $\{\phi_i\}_{i=0}^N$ be the set of hat functions such that $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ for $j \ne i$. The finite volume approximation u_h to the true solution u of problem (4) can be expressed as

$$u_h(x) := \sum_{j=1}^{N-1} u_j \phi_j(x) + u_l \phi_0(x) + u_r \phi_N(x).$$