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Superconvergence property of an over-penalized discontinuous Galerkin finite element gradient recovery method

Lunji Song [a,*], Zhimin Zhang [b,c]

[a] School of Mathematics and Statistics, and Key Laboratory of Applied Mathematics and Complex Systems (Gansu Province), Lanzhou University, Lanzhou 730000, PR China
[b] Beijing Computational Science Research Center, Beijing 100084, PR China
[c] Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

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A B S T R A C T

A polynomial preserving recovery method is introduced for over-penalized symmetric interior penalty discontinuous Galerkin solutions to a quasi-linear elliptic problem. As a post-processing method, the polynomial preserving recovery is superconvergent for the linear and quadratic elements under specified meshes in the regular and chevron patterns, as well as general meshes satisfying *Condition* (ϵ, σ). By means of the averaging technique, we prove the polynomial preserving recovery method for averaged solutions is superconvergent, satisfying similar estimates as those for conforming finite element methods. We deduce superconvergence of the recovered gradient directly from discontinuous solutions and naturally construct an *a posteriori* error estimator. Consequently, the *a posteriori* error estimator based on the recovered gradient is asymptotically exact. Extensive numerical results consistent with our analysis are presented.

1. Introduction

In recent years there have been superconvergence results of the gradient and gradient recovery schemes [9,12,20,23,31–33,36–38], while their contributions are based on finite element approximations and attract many researchers from the fields of modern engineering and scientific computation. The Zienkiewicz–Zhu (ZZ) error estimator [38] is referred to the superconvergence patch recovery (SPR), which is based on gradient recovery from the gradient of the finite element solution on patches in the discrete least-squares fitting sense. The robustness of the ZZ patch recovery is originated from its superconvergence under structured meshes. As a new strategy, polynomial preserving recovery (PPR) has first been introduced by Zhang in [24,34] with the use of the fitted finite element solution values to recover the gradient. The PPR keeps all known superconvergence properties of the ZZ patch recovery, out-performing the SPR in the cases of quadratic element at edge centers and linear element for the chevron mesh [36]. The PPR has superconvergence in mildly structured grids as well as anisotropic grids [34,35]. Therefore, for gradient recovery to finite element solutions, the PPR method is a good alternative.

* Corresponding author.
E-mail addresses: song@lzu.edu.cn (L. Song), zzhang@math.wayne.edu (Z. Zhang).

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It is well known that if the recovered quantity better approximates the exact one, then it can be used in constructing asymptotically exact *a posteriori* error estimates (see [1]). The PPR becomes standard in finite element methods and has been adopted in some commercial softwares (COMSOL etc.) as a superconvergence tool.

We consider the following second-order quasi-linear elliptic problem

$$\begin{cases} -\nabla\cdot(a(x,u)\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded convex domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, and $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$. We assume $0 < a_1 \le a(x,v) \le a_2$, $x \in \bar{\Omega}$, $v \in \mathbb{R}$ for some positive constants a_1, a_2 and $a(x,v) \in C_b^2(\bar{\Omega}\times\mathbb{R})$, where $C_b^2(\bar{\Omega}\times\mathbb{R})$ is the space of twice continuously differentiable functions on $\mathbb{R}$ whose first and second order derivatives are bounded in $\bar{\Omega}\times\mathbb{R}$. It holds from [15] that there exists a unique weak solution u to (1) and $u \in C^{2+\delta}$ with $\delta \in (0,1)$ when $f \in C^{\delta}(\Omega)$ and the boundary $\partial\Omega$ is smooth. The equation (1), supplemented with the homogeneous Dirichlet boundary condition, describes an equilibrium state of a chemical species of the concentration u in a porous medium with a source term $f(x)$.

Interior penalty discontinuous Galerkin (IPDG) methods are a powerful simulation tool for solving linear or nonlinear equations (see e.g. [4,11,14,17,18,22,25,28]). There are some primal DG versions belonged to IPDG methods (see [3,19]), such as symmetric interior penalty Galerkin (SIPG), nonsymmetric interior penalty Galerkin (NIPG), incomplete interior penalty Galerkin (IIPG) as well as its corresponding over-penalized interior penalty methods. We are interested in an over-penalized symmetric interior penalty Galerkin (OPSIPG) method [27,30] to realize a gradient recovery. One of reasons is that its penalty parameters can be bounded above rather than sufficiently large in the usual SIPG method for refined grids. The OPSIPG method we use preserves the integral terms on hybrid multiplication of jump or average of test and trial functions on interior edges. The weakly over-penalized symmetric interior penalty method (WOPSIP) presented by Brenner in [6] ignores the hybrid multiplication terms. The OPSIPG and WOPSIP methods produce an ill-conditioned discrete system, which results from the over-penalization terms. Fortunately, it can be remedied by a simple block-diagonal preconditioner (see [6]) and a multilevel preconditioner in [8]. Now the main question lies in how to implement the PPR technique into discontinuous Galerkin solutions under the framework of discontinuous Galerkin finite element methods.

In the present work, we aim to the PPR technique based on discontinuous Galerkin solutions and its theoretical analysis. The PPR technique is good for arbitrary order Lagrange finite elements, then for simplicity, we would focus on the linear and quadratic elements, which are widely used in practice. Several steps for the PPR are needed: we choose a patch including necessary or enough points first, and then by the fitted solution values recover the gradient, and further construct an *a posteriori* error estimate in the energy norm. Due to the PPR partial to the symmetry of patches, we shall consider some specified meshes in the regular and chevron patterns, as well as general meshes satisfying *Condition* (ϵ,σ). In case that the resultant DG system becomes ill-conditioned from over-penalized parameters, it is important to use a simple block-diagonal preconditioner remedying the problem. To the best of our knowledge, this is the first theoretical superconvergence proof for the PPR implemented on discontinuous Galerkin solutions to nonlinear elliptic problems. Furthermore, the proposed method can be used to solve time-dependent diffusion problems, e.g., the problem discussed in [13]. Our method can be applied to the spatial discretization part at each time level while maintaining the time discretization part unchanged.

The remainder of this paper is organized as follows. In Section 2 we introduce the over-penalized interior penalty discontinuous Galerkin (OPIPDG) formulas in the broken Sobolev space to approximate elliptic equations. In Section 3, we state and prove some preliminary lemmas for OPSIPG scheme analogous to those appear in the usual SIPG method. In Section 4, the gradient recovery operator will be constructed for OPSIPG solutions, thereafter we prove the main results for the gradient recovery, which can be used to define an *a posteriori* error estimator. In the last section, several numerical examples are given to illustrate superconvergence of the gradient recovery for linear and quadratic elements in some structured meshes as well as unstructured meshes, and also show that an *a posteriori* error estimator is asymptotically exact for a corner singularity problem.

2. The over-penalized discontinuous Galerkin method

Let $\mathcal{E}_h$ be a subdivision of Ω into disjoint open elements such that $\bar{\Omega} = \bigcup_{i=1}^{N_h} \bar{E}_i$, where E_i is a triangle in 2D and N_h is the number of all elements. We let $h_k := diam(\bar{E}_k)$ and $h := \max_{E\in\mathcal{E}_h} h_k$. It is assumed that the family of subdivisions $\mathcal{E}_h$ is shape regular and each element $E \in \mathcal{E}_h$ shall be an affine image of a standard reference element. Assume that the mesh $\mathcal{E}_h$ is quasi-uniform: for all $E_i \in \mathcal{E}_h$, there exists a constant $\tau > 0$, independent of h, such that

$$h \le \tau h_i, \tag{2}$$

where ρ_i denotes the diameter of the largest circle inscribed in E_i. We introduce the set of all edges of the mesh $\mathcal{E}_h$ by

$$\mathcal{F}_h := \{e_1, e_2, \cdots, e_{P_h}, e_{P_h+1}, \cdots, e_{M_h}\},$$

where

$$\begin{cases} e_i \subset \Omega, & \text{if } 1 \le i \le P_h, \\ e_i \subset \partial\Omega, & \text{if } P_h + 1 \le i \le M_h. \end{cases}$$