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Three-point combined compact difference schemes for time-fractional advection–diffusion equations with smooth solutions ☆

Guang-Hua Gao [a], Hai-Wei Sun [b,*]

[a] College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, PR China
[b] Department of Mathematics, University of Macau, Macao

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ABSTRACT

In this paper, we study the numerical solution for the time-fractional advection–diffusion equation (TFADE). A high-order accurate three-point combined compact difference scheme with the *L1 formula* is proposed to solve a class of TFADEs. The method is globally $(2-\gamma)$th-order accurate in time and at least fifth-order accurate in space for the constant coefficient TFADEs subject to periodic boundary conditions, where γ is the order of time-fractional derivative in the governing equation. The unconditional stability and the high-order convergence are proved using the Fourier analysis method. The proposed method can be extended to treat the variable coefficient TFADEs subject to other local boundary conditions. Several numerical examples are computed to validate the numerical accuracy, effectiveness and robustness of the present method for TFADEs with smooth solutions and different boundary conditions. Moreover, the proposed algorithm preserves the higher order accuracy for advection-dominated problems with the smooth solutions.

1. Introduction

The fractional operator is non-local and it considers the historic distributed effects. Numerous applications and physical manifestations promote the development of fractional calculus in the past few decades. Various fractional partial differential equations have been established to depict phenomenons in fields of engineering, science and economics, such as carrier transport in amorphous semiconductors, system identification and control, anomalous diffusion in electrochemistry, fractance circuits, electrode electrolyte interface, viscoelasticity, fractional neural modeling in bio-sciences, chaos theory, finance and so on [1–6].

For most time-fractional advection–diffusion equations (TFADEs), it is not an easy task to seek for their analytical solutions. Although they are available for some simple cases, the solutions often refer to some special functions, which are quite sophisticated in calculation. Therefore, it is essential to survey the effective numerical solutions of TFADEs. Recently, a tremendous amount of works were devoted to the finite difference methods for solving the fractional differential

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* Corresponding author.
E-mail addresses: gaoguanghua1107@163.com (G.-H. Gao), HSun@umac.mo (H.-W. Sun).

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equations. Yuste and Acedo [7] established an explicit difference scheme for solving the fractional sub-diffusion equations, with the Grünwald–Letnikov approximation for the Riemann–Liouville time-fractional derivative and a forward Euler approximation for the first-order time derivative. The von Neumann-type analysis is applied to determine stability conditions of the obtained difference scheme. For the same fractional sub-diffusion equations with Neumann boundary conditions (NBCs), Langlands and Henry [8] presented an implicit difference scheme with the *L1* approximation for the Riemann–Liouville time-fractional derivative and a backward Euler approximation for the first-order time derivative. Although the accuracy to approximate the time-fractional derivative was improved from order one to order $2-\gamma$ for the Riemann–Liouville fractional derivative of order γ $(0<\gamma<1)$, the whole numerical accuracy of difference scheme still kept order one due to the first-order Euler approximation for the first-order time derivative. Liu [9] also investigated an implicit difference approach by combining the *L1 formula* for the Caputo time-fractional derivative of order γ and the second-order central difference quotient for the second-order space derivative. Nevertheless, only the first-order convergence in time was proved by the discrete maximum principle. Sun and Wu [10] discussed the unconditionally stable difference schemes for fractional diffusion-wave equations using the *L1 formula* to approximate the Caputo time-fractional derivatives. The rigorous proof for numerical accuracy of the *L1 formula* was given in [10]. The proof of this formula was also drawn by another way during the discussion of spectral methods for the fractional sub-diffusion [11] and the fractional cable equations [12]. For the fractional sub-diffusion problem, Chen et al. [13] developed an implicit difference scheme using the first-order Grünwald–Letnikov approximation for the Riemann–Liouville time-fractional derivative and analyzed the scheme by the Fourier method. An implicit moving least squares meshless method was used to solve the time-dependent fractional advection–diffusion equations in [14]. About more relevant works on the subject, readers can refer to the literature [15–17] and references therein.

As we have mentioned above, the fractional operator is non-local and with memory of history. For the time-fractional differential equations, when computing the values of unknowns on the current time level, all function values of unknowns on previous time levels need to be stored. Therefore, it is important and meaningful to develop some higher-order accurate numerical methods for solving the time fractional differential equations. In 2009, Cui [18] constructed a compact difference scheme with the first-order accuracy in time and fourth-order accuracy in space for solving a fractional sub-diffusion equation. Later, Du, Cao and Sun [19], Gao, Sun and Zhang [20], Hu and Zhang [21,22], Mohebbi and Abbaszadeh [23], and Ren, Sun and Zhao [24] studied the spatial fourth-order compact schemes for solving several types of time-fractional partial differential equations. All above compact difference schemes are only constructed for Dirichlet boundary value problems (except [24] which considered the NBCs) and achieve the fourth-order numerical accuracy in space. Recently, some new works on the higher-order numerical approximation for the time-fractional derivatives can also be found. Cao and Xu [25] obtained a higher-order scheme for the numerical solution of the fractional ordinary differential equations starting from the equivalent integral form of the original differential equations. Gao and Sun [26] proposed a new *L1–2 formula* to approximate the Caputo time-fractional derivatives and investigated the applications of this formula into solving fractional differential equations. Zeng et al. [27] presented the fractional linear multistep method for the discretization of the time-fractional derivatives. Li and Ding [28] used the Grünwald formula with the coefficients generated from the quadratic polynomial to improve the numerical accuracy for approximating the time-fractional derivatives. Wang and Vong [29] developed the high-order difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation using the weighted and shifted Grünwald formula.

In 1998, Chu and Fan [30] proposed a combined compact difference (CCD) method for solving 1D and 2D steady convection diffusion equations. The CCD method in [30] is an implicit three-point scheme, with the sixth-order accuracy of local truncated approximation, which can be efficiently solved by the so-called triple-tridiagonal solver [30]. In related numerical methods for solving the partial differential equations by the CCD scheme, the first- and second-order derivatives together with the function values of unknowns at grid points are computed simultaneously. More developments on the CCD method can be found in [31–35]. Nevertheless, to our knowledge, the three-point CCD scheme has never been employed to solve the time-fractional partial differential equations before. Moreover, the global convergence analysis of the CCD scheme was not studied in the literature.

In this paper, we propose the three-point CCD scheme along with the *L1 formula* for solving the 1D TFADEs. The choice of the *L1 formula* for approximating the time-fractional derivatives is due to the good properties of coefficients in this formula which facilitate the theoretical analysis on the corresponding schemes. At first, we investigate the constant coefficient TFADE subject to the periodic boundary condition (PBC) using the Fourier analysis method. For this case, the unconditional stability, the global at least fifth-order accuracy in space and the global $(2-\gamma)$th-order accuracy in time of the proposed method are theoretically proved. Then we extend the proposed method to the variable coefficient cases with other local boundary conditions. As we know, it is quite complicated to apply the fourth-order Padé-type compact technique for fractional diffusion or cable equations [18–22] to construct the compact difference scheme for variable coefficient and advection–diffusion equations, especially for the non-self-adjoint cases. Here, the CCD scheme will be evolved from attempts to alleviate these disadvantages and to provide a perspective for the establishment of some higher-order accurate difference schemes for solving some variable coefficient and advection–diffusion equations. We also show the $(2-\gamma)$th-order local accuracy in time and the higher-order local accuracy in space of the proposed methods. We remark that for derivative boundary value problems, such as the problems with NBCs or mixed boundary conditions, we do not need to especially discretize the derivative boundary conditions anymore, which is usually the one of difficult points by the general compact finite difference methods for these problems in which the space derivatives are directly discretized using the difference quotient. Numerical