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Spectral collocation for multiparameter eigenvalue problems arising from separable boundary value problems

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ABSTRACT

In numerous science and engineering applications a partial differential equation has to be solved on some fairly regular domain that allows the use of the method of separation of variables. In several orthogonal coordinate systems separation of variables applied to the Helmholtz, Laplace, or Schrödinger equation leads to a multiparameter eigenvalue problem (MEP); important cases include Mathieu's system, Lamé's system, and a system of spheroidal wave functions. Although multiparameter approaches are exploited occasionally to solve such equations numerically, MEPs remain less well known, and the variety of available numerical methods is not wide. The classical approach of discretizing the equations using standard finite differences leads to algebraic MEPs with large matrices, which are difficult to solve efficiently.
The aim of this paper is to change this perspective. We show that by combining spectral collocation methods and new efficient numerical methods for algebraic MEPs it is possible to solve such problems both very efficiently and accurately. We improve on several previous results available in the literature, and also present a MATLAB toolbox for solving a wide range of problems.

1. Introduction

When we apply separation of variables to the Helmholtz equation of an elliptic membrane, we get Mathieu's system. This two-parameter eigenvalue problem (2EP) is often used as a motivation for the introduction of multiparameter eigenvalue problems (MEPs), see, e.g., [40]. Yet, it was not until [16] that this approach was actually used to compute the eigenfrequencies of an elliptic membrane. In [16] it is shown that the two-parameter approach has certain advantages with respect to the accuracy as well as to the required computational time and can be used in practice to numerically evaluate a large number of eigenfrequencies.

Compared with [16], this paper presents advances in several directions. First, we consider several other very important problems. In addition to Mathieu's system, discretization with spectral collocation in conjunction with numerical methods

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for the obtained algebraic MEP can be applied to Lamé's system as well as other MEPs that arise by separation of variables. To the best of our knowledge, this technique has never been applied to these problems until now. In particular, in Example 6 we apply this method to a three-parameter eigenvalue problem (3EP) and compute eigenmodes of a challenging ellipsoidal wave equation.

Second, while a Jacobi–Davidson style solver [18,19] and standard implicitly restarted Arnoldi [36] were used in [16] to solve the 2EPs, we speed up the computations in this paper even more by exploiting a new fast Sylvester–Arnoldi algorithm for 2EPs from [28]; this algorithm is briefly described in Section 6. Third, we compare with and improve on several available results in the literature in the numerical examples in Section 7. Finally, we present a MATLAB toolbox for solving MEPs coming from various equations.

The rest of this paper is organized as follows. Section 2 provides several examples of the technique of separation of variables leading to MEPs. A generic form of such MEP is treated in Section 3. In Section 4 we give an overview of the spectral collocation method, which is used to discretize a MEP into an algebraic MEP. This eigenvalue problem is presented in Section 5. In Section 6 we give an overview of available numerical methods for algebraic MEPs with an emphasis on the recent Sylvester–Arnoldi method from [28]. An important part of the paper are the numerical examples in Section 7. In several examples we demonstrate that spectral collocation combined with the Sylvester–Arnoldi method can compute several hundreds of the smallest eigenmodes very efficiently and accurately; we hereby improve various previous results. In Appendix A we describe the main functions in a freely available MATLAB toolbox MultiParEig that contains the implementations of all algorithms and numerical examples from this paper.

2. Motivating problems

Whenever the separation of variables is used to solve a boundary value problem related to a PDE, a system of ODEs is obtained in the first instance. Then, the boundary conditions of the problem at hand dictate boundary conditions for the unknowns of the systems of ODEs involved. Consequently, a MEP is now well defined.

In this section we give some examples of boundary value problems where separation of variables leads to MEPs. We do not attempt to describe all possible situations; for a good overview of all possible coordinate systems and related boundary value problems, see, e.g., [24,29,31,45]. Additional examples together with numerical solutions can be found in Section 7.

2.1. Mathieu's system

This is probably the most well-known example of a 2EP. Separation of variables applied to the two-dimensional Helmholtz equation $\nabla^2 u + \omega^2 u = 0$ in elliptic coordinates

$$x = h\,\cosh(\xi)\,\cos(\eta),$$
$$y = h\,\sinh(\xi)\,\sin(\eta),$$

where $0 \le \xi < \infty$ and $0 \le \eta < 2\pi$, leads to $u = G(\eta)\,F(\xi)$, where G and F satisfy the coupled system of Mathieu's angular and radial equations (for details, see, e.g., [40])

$$\begin{aligned} G''(\eta) + (\lambda - 2\mu\,\cos(2\eta))\,G(\eta) &= 0, \\ F''(\xi) - (\lambda - 2\mu\,\cosh(2\xi))\,F(\xi) &= 0. \end{aligned} \tag{1}$$

The parameter μ is related to the eigenfrequency ω by $\mu = \frac{1}{4}h^2\omega^2$, where $h = \sqrt{\alpha^2 - \beta^2}$ with $\alpha = h\cosh(\xi_0)$ (the major axis) and $\beta = h\sinh(\xi_0)$ (the minor axis of the membrane), and λ is a separation constant. The problem along with the appropriate boundary conditions is treated as a 2EP in [16] and solved numerically very accurately and efficiently with the Chebyshev collocation. As discussed in the introduction, we will extend the results in [16] considerably in several ways.

2.2. Lamé's system

When separation of variables is applied to the three-dimensional Helmholtz equation $\nabla^2 u + \omega^2 u = 0$ in sphero-conal coordinates

$$x = r\,\cos(\varphi)\,(1 - k'^2\cos^2(\theta))^{1/2},$$
$$y = r\,\cos(\theta)\,(1 - k^2\cos^2(\varphi))^{1/2},$$
$$z = r\,\sin(\theta)\,\sin(\phi),$$

where $r \ge 0$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, $0 \le k, k' \le 1$, and $k^2 + k'^2 = 1$, it gives $u = R(r)\,L(\varphi)\,N(\theta)$, where R, L, and N satisfy the system of differential equations

$$r^2 R''(r) + 2r\,R'(r) + [\omega^2 r^2 - \rho\,(\rho+1)]\,R(r) = 0, \tag{2}$$

$$(1 - k^2\cos^2(\varphi))\,L''(\varphi) + k^2\,\sin(\varphi)\,\cos(\varphi)\,L'(\varphi) + [k^2\,\rho\,(\rho+1)\sin^2(\varphi) + \delta]\,L(\varphi) = 0, \tag{3}$$

$$(1 - k'^2\cos^2(\theta))\,N''(\theta) + k'^2\,\sin(\theta)\,\cos(\theta)\,N'(\theta) + [k'^2\,\rho\,(\rho+1)\sin^2(\theta) - \delta]\,N(\theta) = 0, \tag{4}$$