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A fast method for a generalized nonlocal elastic model

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ABSTRACT

We develop a numerical method for a generalized nonlocal elastic model, which is expressed as a composition of a Riesz potential operator with a fractional differential operator, by composing a collocation method with a finite difference discretization.

By carefully exploring the structure of the coefficient matrix of the numerical method, we develop a preconditioned fast Krylov subspace method, which reduces the computations to $(N \log N)$ per iteration and the memory to $O(N)$. The use of the preconditioner significantly reduces the number of iterations, and the preconditioner can be inverted in $O(N \log N)$ computations. Numerical results show the utility of the method.

1. Introduction

In recent years many nonlocal models are emerging as a powerful tool for modeling challenging phenomena in a variety of disciplines, including anomalous transport, and long range time memory or spatial interactions. In the context of diffusion processes, traditional second-order diffusion equations model Fickian diffusion processes, in which a particle's movement obeys a Brownian motion. However, in the past few decades, many diffusion processes were found to exhibit anomalous diffusion behavior, in which the probability density functions of the underlying particle motions are characterized by an algebraically decaying tail and so cannot be modeled properly by second-order diffusion equations. Fractional diffusion equations provide an adequate and accurate description of transport processes that exhibit anomalous diffusion, as the probability density functions of anomalous diffusion processes satisfy fractional diffusion equations [2,21].

In the context of solid mechanics, classical theory assumes that all internal forces in a body act through a zero distance, which leads to mathematical models described by partial differential equations. These models do not provide a proper description of problems with spontaneous formation of discontinuities. In the last decade or so, the peridynamic theory was proposed as a reformulation of solid mechanics [8,25], and provides an appropriate description of the deformation of a continuous body involving discontinuities or other singularities, which cannot be described properly by the classical partial differential equation models. Mathematically, the peridynamic models are expressed as an integral equation in terms of a Riesz potential operator, which does not explicitly involve the notion of deformation gradients.

Continuum elastic models have been widely used to study the dynamics of real physical systems in such applications as flexible polymers, growing interfaces, and membranes [7,18,37]. Recently, a generalized elastic model was developed in [27] to study a generalization and relations of these elastic models. This model is expressed as a composition of a Riesz-like

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potential operator with a fractional differential operator [1,24]. In a sense, the generalized elastic model can be viewed as a composition of a peridynamic model with a fractional differential equation model.

Due to the nonlocal nature and complexity of these nonlocal models, the corresponding numerical methods typically generate dense or full coefficient matrices. Traditionally, direct solvers were used to solve the numerical schemes, which have $O(N^3)$ computational complexity and require $O(N^2)$ storage for a numerical discretization of N unknowns [8,17,20]. Extensive research has been conducted on the development of fast and accurate numerical methods for nonlocal models. In the context of fractional differential equations, fast operator-splitting, or alternating-direction, or Krylov subspace-based numerical methods were developed with significantly reduced computational complexity and storage requirement [5,15,19,20,28,30,33,34,36]. In the context of peridynamic or nonlocal diffusion models, because the corresponding integral operator is fully coupled in all the directions, alternating-direction approach does not seem to be very efficient. A fast Krylov subspace-based method was developed that has an optimal order storage and an almost linear computational complexity [31,32]. Numerical experiments show the utility of these fast methods.

However, due to the complexity of the generalized elastic model, its numerical discretization requires extra care. The finite volume methods [16,29,35] and aforementioned finite difference methods are well suited for the internal fractional differential operators in the model. However, they do not conveniently discretize the external Riesz potential operator. On the other hand, a collocation method is well fit for the external Riesz potential operator [11], but is inconvenient to discretize the internal fractional differential operator. A Galerkin method can be applied to discretize both the external Riesz potential operator and the internal fractional differential operator [9,12,13], but the computational cost is increased significantly as a double integral has to be used to discretize the external Riesz potential operator.

The objective of this paper is two folds: (i) we design a feasible numerical method for the model by composing a collocation method, which is well fit for the external Riesz potential operator, with a finite difference method, which is well suited for the discretization of the internal fractional differential operator. (ii) By carefully exploring the structure of the coefficient matrix of the numerical method, we develop a preconditioned fast Krylov subspace method, which reduces the computations to $(N \log N)$ per iteration and the storage to $O(N)$. The use of the preconditioner significantly reduces the number of iterations, and the preconditioner can be inverted in $O(N \log N)$ computations.

2. Model problem

We consider the following nonconventional Dirichlet boundary-value problem of the generalized fractional elastic model

$$\begin{aligned}&\int_0^1 \Lambda(x-y)\Big(d^+(y)\frac{\partial^\beta u(y)}{\partial_+ y^\beta} + d^-(y)\frac{\partial^\beta u(y)}{\partial_- y^\beta}\Big)dy = f(x), \quad x \in (0,1),\\&u(x) = 0, \quad x \notin (0,1).\end{aligned} \tag{1}$$

Here $d^+(x)$ and $d^-(x)$ are the left-sided and right-sided diffusivity coefficients, $\frac{\partial^\beta u}{\partial_+ x^\beta}$ and $\frac{\partial^\beta u}{\partial_- x^\beta}$ are the left-sided and right-sided (Grünwald–Letnikov) fractional derivatives of order β $(1 < \beta < 2)$ [22,24]

$$\begin{aligned}\frac{\partial^\beta u}{\partial_+ x^\beta} &= \lim_{h\to 0}\frac{1}{h^\beta}\sum_{l=0}^{\lfloor x/h\rfloor} g_l^{(\beta)} u(x-lh),\\ \frac{\partial^\beta u}{\partial_- x^\beta} &= \lim_{h\to 0}\frac{1}{h^\beta}\sum_{l=0}^{\lfloor (1-x)/h\rfloor} g_l^{(\beta)} u(x+lh)\end{aligned} \tag{2}$$

where $\lfloor x \rfloor$ represents the floor of x and $g_k^{(\beta)} = (-1)^k\binom{\beta}{k}$ with $\binom{\beta}{k}$ being the fractional binomial coefficients. It is known that the Grünwald–Letnikov fractional derivatives coincide with the Riemann–Liouville fractional derivatives defined by [22,24]

$$\begin{aligned}{}_0^R D_x^\beta u &:= \frac{1}{\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_0^x \frac{u(s)}{(x-s)^{\beta-1}}ds,\\ {}_x^R D_1^\beta u &:= \frac{1}{\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_x^1 \frac{u(s)}{(s-x)^{\beta-1}}ds.\end{aligned} \tag{3}$$

An arithmetic average of the left and right Riemann–Liouville fractional derivatives with a proper normalization constant gives a Riesz fractional derivative [24].

In addition, $\Lambda(x)$ represents the long-range hydrodynamic interactions, which is of the form

$$\Lambda(x) = \frac{1}{|x|^\alpha}, \quad 0 < \alpha < 1 \tag{4}$$