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Compact difference scheme for distributed-order time-fractional diffusion-wave equation on bounded domains

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ABSTRACT

In this paper, we derive and analyse a compact difference scheme for a distributed-order time-fractional diffusion-wave equation. This equation is approximated by a multi-term fractional diffusion-wave equation, which is then solved by a compact difference scheme. The unique solvability of the difference solution is discussed. Using the discrete energy method, we prove the compact difference scheme is unconditionally stable and convergent. Finally, numerical results are presented to support our theoretical analysis.

1. Introduction

An important application of distributed-order equations is to model ultraslow diffusion where a plume of particles spreads at a logarithmic rate [1–3]. When the order of the fractional derivative is distributed over the unit interval, it is useful for modeling a mixture of delay sources [4]. Also, distributed-order equations may be viewed as consisting of viscoelastic and visco-inertial elements when the order of the fractional derivative varies from zero to two [5,6]. Motivated by these applications, some attention has been paid to the fractional partial differential equations (FPDEs) with distributed order [7–10].

Chechkin et al. [11] proposed diffusionlike equations with time fractional derivatives of the distributed order for the kinetic description of anomalous diffusion and relaxation phenomena and proved the positivity of the solutions of their proposed equations. They demonstrated that retarding subdiffusion and accelerating superdiffusion were governed by distributed-order fractional diffusion equations [12]. The fundamental solutions for the one-dimensional time fractional diffusion equation and multi-dimensional diffusion-wave equation of distributed order were obtained by Mainardi et al. [13,14] and Atanackovic et al. [15], respectively. Atanackovic et al. [16] also proved the existence of the solution to the Cauchy problem for the time distributed order diffusion equation and calculated it by the use of Fourier and Laplace transformations. Furthermore, they studied waves in a viscoelastic rod of finite length, where viscoelastic material was described by a constitutive equation of fractional distributed-order type (see [17]). Luchko [18] proved the uniqueness and continuous dependence on initial conditions for the generalized time-fractional diffusion equation of distributed order on bounded domains. Meerschaert et al. [4] provided explicit strong solutions and stochastic analogues for distributed-order time-fractional diffusion equations on bounded domains, with Dirichlet boundary conditions.

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On the other hand, different numerical methods for solving FPDEs have been proposed [19–22]. Recently, Liu et al. [23] proposed some computationally effective numerical methods for simulating the multi-term time-fractional diffusion-wave equations. There are also some papers discussing numerical methods of the distributed-order equations. For example, Diethelm and Ford [24] developed a numerical scheme for the solution of a distributed-order Fractional ordinary DE and gave a convergence theory for their method. Based on the matrix form representation of discretized fractional operators (see [25]), Podlubny et al. [26] extended the range of applicability of the matrix approach to discretization of distributed-order derivatives and integrals, and to numerical solution of distributed-order differential equations (both ordinary and partial). Katsikadelis [27] presented an efficient numerical method to solve linear and nonlinear distributed-order FODEs. However, published papers on numerical methods of the distributed-order FPDEs are sparse. This motivates us to consider effective numerical methods for distributed-order time-fractional diffusion-wave equations.

In this paper, we first approximate the integral term in the distributed-order diffusion-wave equation using numerical approximation. Then the given distributed-order equation can be written as a multi-term time fractional diffusion-wave equation. We derive a compact difference scheme which is uniquely solvable for the multi-term fractional diffusion-wave equation. Using the discrete energy method, we prove the compact difference scheme is unconditionally stable and convergent. Finally, two numerical examples are provided to show the effectiveness of our method.

The rest of the paper is organized as follows. In Section 2, a compact difference scheme is derived. Section 3 presents the solvability, stability and convergence for the compact difference scheme. Two examples are given in Section 4 and some conclusions are drawn in Section 5.

2. Compact difference scheme

Consider the following distributed-order time-fractional diffusion-wave equations

$$\mathbb{D}_t^{\varpi(\alpha)} u(x,t) = K\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \tag{2.1}$$

in an open bounded domain $0<x<L, 0<t<T$. Here $K>0$, x and t are the space and time variables. The time-fractional derivative $\mathbb{D}_t^{\varpi(\alpha)}$ of distributed order is defined by

$$\mathbb{D}_t^{\varpi(\alpha)} u(x,t) = \int_1^2 {}_0^cD_t^{\alpha} u(x,t)\varpi(\alpha)d\alpha \tag{2.2}$$

with the left-side Caputo fractional derivative ${}_0^cD_t^{\alpha}$ defined as (see [28])

$${}_0^cD_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{\partial^n u}{\partial \tau^n}(x,\tau)d\tau, & n-1<\alpha<n, \\ \frac{\partial^n u}{\partial t^n}(x,t) & , \quad \alpha = n. \end{cases} \tag{2.3}$$

and with a continuous non-negative weight function $\varpi:[1,2]\to\mathcal{R}$ that is not identically equal to zero on the interval $[1,2]$, such that the conditions

$$0\le\varpi(\alpha), \varpi\ne 0, \alpha\in[1,2], \int_1^2 \varpi(\alpha)d\alpha = W > 0 \tag{2.4}$$

hold true, where W is a positive constant.

In this paper, the initial-boundary conditions

$$u(x,0)=\phi_1(x), \quad u_t(x,0)=\phi_2(x), \qquad 0\le x\le L, \tag{2.5}$$

$$u(0,t)=\psi_1(t), \quad u(L,t)=\psi_2(t), \qquad 0\le t\le T \tag{2.6}$$

for Eq. (2.1) are considered.

Now, we state our numerical method as follows.

Step 1: Discretize the integral term in the distributed-order equation.

Let us discretize the interval $[1,2]$, in which the order α is changing, using the grid $1=\xi_0<\xi_1<\cdots<\xi_q=2 (q\in\mathcal{N})$, with the steps $\Delta\xi_s$ not necessarily equidistant. We obtain

$$\mathbb{D}_t^{\varpi(\alpha)} u(x,t) \approx \sum_{s=1}^{q}\varpi(\alpha_s)\left({}_0^cD_t^{\alpha_s}u(x,t)\right)\Delta\xi_s = \sum_{s=1}^{q} d_s\, {}_0^cD_t^{\alpha_s}u(x,t), \tag{2.7}$$

where $\alpha_s\in(\xi_{s-1},\xi_s]$, $d_s=\varpi(\alpha_s)\Delta\xi_s$, $\Delta\xi_s=\xi_s-\xi_{s-1}, s=1,2,\cdots,q$.