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A moving mesh finite difference method for equilibrium radiation diffusion equations ☆

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ABSTRACT

An efficient moving mesh finite difference method is developed for the numerical solution of equilibrium radiation diffusion equations in two dimensions. The method is based on the moving mesh partial differential equation approach and moves the mesh continuously in time using a system of meshing partial differential equations. The mesh adaptation is controlled through a Hessian-based monitor function and the so-called equidistribution and alignment principles. Several challenging issues in the numerical solution are addressed. Particularly, the radiation diffusion coefficient depends on the energy density highly nonlinearly. This nonlinearity is treated using a predictor–corrector and lagged diffusion strategy. Moreover, the nonnegativity of the energy density is maintained using a cutoff method which has been known in literature to retain the accuracy and convergence order of finite difference approximation for parabolic equations. Numerical examples with multi-material, multiple spot concentration situations are presented. Numerical results show that the method works well for radiation diffusion equations and can produce numerical solutions of good accuracy. It is also shown that a two-level mesh movement strategy can significantly improve the efficiency of the computation.

1. Introduction

Radiation diffusion plays an important role in a variety of physical applications such as inertially confined fusion, combustion simulation, and atmospheric dynamics. When photon mean free paths are much shorter than characteristic length scales, a diffusion approximation can be used to describe the radiation penetrating from a hot source to a cold medium. This diffusion approximation forms a highly nonlinear diffusion coefficient and gives a sharp hot wave steep front (often referred to as a Marshak wave). Solutions near this steep front can vary dramatically in a very short distance. Such complex local solution structures make radiation diffusion an excellent example for using mesh adaptation methods because the number of mesh points can be prohibitively large when a uniform mesh is used.

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Generally speaking, there are two types of radiation diffusion equations, equilibrium and non-equilibrium systems. When the energy density E satisfies the relation $E = T^4$, where T is temperature, the system is called in an equilibrium state and otherwise in a non-equilibrium state. Radiation diffusion has attracted considerable attention from researchers in the past; e.g., see [3,16,21,23–25,28–30]. For example, foundations of radiation hydrodynamics can be found in the book [21] while numerical techniques for radiation diffusion and transport are addressed systematically in the book [3]. Rider et al. [25] study multi-material equilibrium radiation diffusion equations and propose a class of algorithms with the Newton–Krylov (GMERS) method preconditioned by a multi-grid method as a special example. Ovtchinnikov and Cai [23] study a parallel one-level Newton–Krylov–Schwarz algorithm for an unsteady nonlinear radiation diffusion problem.

On the other hand, there are only a few published works that have employed mesh adaptation for the numerical solution of radiation diffusion equations. Winkler et al. [26,27] use an adaptive moving mesh method to solve radiation diffusion and fluid equations in one dimension. Unfortunately, their method has difficulty with mesh crossing and cannot be extended in multi-dimensions. Lapenta and Chacón [16] study an equilibrium radiation diffusion equation in one dimension using an adaptive moving mesh method. Pernice and Philip [24] use the AMR method (a structured adaptive mesh refinement method) to solve two-dimensional equilibrium radiation diffusion equations. They employ a fully implicit scheme to integrate the partial differential equations (PDEs), JFNK (Jacobian-free Newton–Krylov) [14,15,22,25] to solve the resulting nonlinear algebraic equations, and the FAC (Fast Adaptive Composite) preconditioner [19,20] to precondition the implicit coefficient matrix. Their numerical results show that the method can capture the fronts of Marshak waves and have good accuracy for problems with smooth initial solutions.

The objective of this paper is to study the finite difference (FD) solution of two-dimensional equilibrium radiation diffusion equations using an adaptive moving mesh method. The method is based on the so-called moving mesh PDE (MMPDE) approach [10] with which the mesh is moved continuously in time using an MMPDE. The latter is defined as the gradient flow equation of a meshing functional formulated based on mesh equidistribution and alignment and taking into full consideration of the shape, size and orientation of mesh elements [7]. The method is combined with treatments of high nonlinearity and preservation of solution nonnegativity of the equations. The high nonlinearity comes from the diffusion coefficient. We use a coefficient-freezing predictor–corrector procedure to linearize the PDEs. More specifically, at the prediction stage the diffusion coefficient is calculated using the energy density at the previous time step while at the correction stage the coefficient is calculated using the energy density obtained at the prediction stage. This predictor–corrector procedure is known to be comparable to the Beam and Warming linearization method in terms of accuracy and stability [17]. It is also easy and efficient to implement since it contains only two steps of the lagged diffusion computation. Note that for each stage we only need to solve linear PDEs so there is no need for nonlinear iteration. Moreover, the procedure allows an easy and effective dealing of negative values occurring in the computed energy density. Radiation diffusion equations admit nonnegative energy densities. It is crucial for numerical approximation to preserve this property. Excessive negative values in the computed solution not only introduce unphysical oscillations but also can cause the computation to exit unexpectedly. We use a cutoff strategy with which negative values in the computed energy density are replaced with zero after each stage. It has been shown in [18] that the cutoff strategy retains the accuracy and convergence order of FD approximation for parabolic PDEs.

The moving mesh method, together with the above described treatments for nonlinearity and preservation of solution nonnegativity, is applied to a two-dimensional equilibrium radiation diffusion equation for two multi-material, multiple spot concentration scenarios. The numerical results show that the method is able to catch interfaces and onsets of new interfaces and concentrate mesh points near them. The results are comparable to those obtained by Pernice and Philip [24] using the AMR method and to those obtained with the uniform mesh of a much larger size. Moreover, it is shown that the computational efficiency can be significantly improved by a two-level mesh movement strategy [6] while maintaining a comparable level of accuracy.

An outline of the paper is as follows. The physical problem and the governing equations are described in Section 2. The moving mesh method and the treatments of nonlinearity and preservation of solution nonnegativity are discussed in Section 3. In Section 4 we present numerical results obtained for two multi-material, multiple spot concentration scenarios. Finally, Section 5 contains conclusions.

2. Problem description

Radiation diffusion occurs in a variety of astrophysical and laboratory settings and can be formulated in a number of forms; e.g., see Mihalas and Mihalas [21]. For a simple setting where the material temperature is in equilibrium with the radiation energy density, radiation diffusion can be modeled by a nonlinear parabolic PDE in dimensionless form as

$$\frac{\partial E}{\partial t} = \nabla \cdot (D_L(E)\nabla E), \tag{2.1}$$

where E is the dimensionless gray radiation energy density and $D_L(E)$ is Larsen's form of the flux-limited diffusion coefficient [1,13] defined as

$$D_L(E) = \left(\frac{1}{D(E)^2} + \frac{|\nabla E|^2}{E^2}\right)^{-\frac{1}{2}}. \tag{2.2}$$