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Solving elliptic problems with discontinuities on irregular domains – the Voronoi Interface Method

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ABSTRACT

We introduce a simple method, dubbed the Voronoi Interface Method, to solve Elliptic problems with discontinuities across the interface of irregular domains. This method produces a linear system that is symmetric positive definite with only its right-hand-side affected by the jump conditions. The solution and the solution's gradients are second-order accurate and first-order accurate, respectively, in the L^∞ norm, even in the case of large ratios in the diffusion coefficient. This approach is also applicable to arbitrary meshes. Additional degrees of freedom are placed close to the interface and a Voronoi partition centered at each of these points is used to discretize the equations in a finite volume approach. Both the locations of the additional degrees of freedom and their Voronoi discretizations are straightforward in two and three spatial dimensions.

1. Introduction

We focus on the class of Elliptic problems that can be written as:

$$\begin{aligned} \underline{\nabla} \cdot (\beta \underline{\nabla} u) + ku = f \qquad & \text{in } \Omega^- \cup \Omega^+, \\ [u] = g \qquad & \text{on } \Gamma, \\ [\beta \underline{\nabla} u \cdot \underline{\boldsymbol{n}}_\Gamma] = h \qquad & \text{on } \Gamma, \end{aligned} \tag{1}$$

where the computational domain Ω is composed of two subdomains, Ω^- and Ω^+, separated by a co-dimension one interface Γ (see Fig. 1), with $\underline{\boldsymbol{n}}_\Gamma$ the outward normal. Here, $\beta = \beta(\underline{\boldsymbol{x}})$, with $\underline{\boldsymbol{x}} \in \mathbb{R}^n$ ($n \in \mathbb{N}$), is bounded from below by a positive constant and $[q] = q_\Gamma^+ - q_\Gamma^-$ indicates a discontinuity in the quantity q across Γ, f is in L^2, g, h and k are given. Note that this general formulation includes possible discontinuities in the diffusion coefficient β and in the gradient of the solution $\underline{\nabla} u$. Dirichlet or Neumann boundary conditions are applied on the boundary of Ω, denoted by $\partial\Omega$. This class of equations, where some or all of the jump conditions are non-zero, is a corner stone in the modeling of the dynamics of

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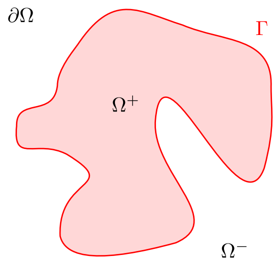

Fig. 1. Geometry of the problem.

important physical and biological phenomena as diverse as multiphase flows with and without phase change, biomolecules' electrostatics, electrokinetics (Poisson–Nerntz–Planck) models with source term or electroporation models.

Given the importance of this class of equations, several approaches have been pursued to computationally approximate their solutions, each with their own pros and cons. The finite element method (FEM) is one of the earliest approaches to solve this problem [4,10,12,19,32,35] and has the advantage of providing a simple discretization formalism that guarantees the symmetry and definite positiveness of the corresponding linear system, even in the case of unstructured grids. It also provides a framework where *a priori* error estimates can be used to best adapt the mesh in order to capture small scale details. However, the FEM is based on the generation of meshes that must conform to the irregular domain's boundary and must satisfy some restrictive quality criteria, a task that is difficult, especially in three spatial dimensions. The difficulty is exacerbated when the domain's boundary evolves during the course of a computation, as it is the case for most of the applications modeled by these equations. Mesh generation is the focus of intense research [58], as the creation of unwanted sliver elements can deteriorate the accuracy of the solution.

Methods based on *capturing* the jump conditions do not depend on the generation of a mesh that conforms to the domain's boundary, hence avoiding the mesh generation difficulty altogether. However, they must impose the boundary conditions implicitly, which is a non-trivial task. A popular approach is the Immersed Interface Method (IIM) of Leveque and Li [36], and the more recent development of Immersed Finite Element Method (IFEM) and Immersed Finite Volume Method (IFVM) [40,27,23]. The basis of the IIM is to use Taylor expansions of the solution on each side of the interface and modify the stencils local to the interface in order to impose the jump conditions. As such, solutions can be obtained on simple Cartesian grids and the solution is second-order accurate in the L^∞ norm. The corresponding linear system, however, is asymmetric unless the coefficient β has no jump across the interface. Another difficulty is the need to approximate surface derivatives along Γ as well as the evaluation of high-order jump conditions. These difficulties have been addressed in the Piecewise-polynomial Interface Method of Chen and Strain [11] and several other approaches have improved the efficacy of the IMM [38,39,64,9,1–3]. We note also that the earliest approach on Cartesian grid is that of Mayo [43], who derived an integral equation to solve the Poisson and the bi-harmonic problems with piecewise constant coefficients on irregular domains; the solution is second-order accurate in the L^∞ norm. We also refer the interested researcher to the matched interface and boundary (MIB) method [66,65].

The finite element community has also proposed embedded interface approaches, including discontinuous Galerkin and the eXtended Finite Element Method (XFEM) [37,29,48,17,8,47,33,22,28,62]. The basic idea is to introduce additional degrees of freedom[1] near the interface and augment the standard basis functions on these elements with basis functions that are combined with a Heaviside function in order to help capture the jump conditions.

In [16], the authors introduce a second-order accurate discretization in the case of, possibly adaptive, Cartesian meshes using a cut-cell approach. The jump condition is imposed by determining the fluxes on both side of the interface, which are constructed from a combination of least squares and quadratic approximations. In [51], the authors also use a cut cell approach but impose the jump with the help of a compact 27-point stencil.

The Ghost Fluid Method (GFM), originally introduced to approximate two-phase compressible flows [21], has been applied to the system problem (1) in [41]. The basic idea is to consider fictitious domains and ghost values that capture the jump conditions in the discretization at grid nodes near the interface. An advantage of this approach is that only the right-hand-side of the linear system is affected by the jump conditions. However, in order to propose a dimension-by-dimension approach, the projection of the normal jump conditions must be projected onto the Cartesian directions. As a consequence, the tangential component of the jump is ignored. Nonetheless, the method has been shown to be convergent with first-order accuracy [42]. The GFM was also shown to produce symmetric positive definite second-order accuracy [25] and even fourth-order accuracy [24], but for a different class of problem, namely for solving Elliptic problems on irregular domains with Dirichlet boundary conditions. In fact, symmetric positive definite second-order accurate solutions can also be obtained in the case where Neumann or Robin boundary conditions are imposed on irregular domains [55,50,53,54]. These methods can be trivially extended to the case of adaptive Cartesian grids and we refer the interested readers to the review of Gibou, Min and Fedkiw [26] for more details. In the case of jump conditions, Coco and Russo [14] have also used a fictitious domain

[1] We understand by degrees of freedom the set of locations at which the solution is sampled.