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A model and variance reduction method for computing statistical outputs of stochastic elliptic partial differential equations

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ABSTRACT

We present a model and variance reduction method for the fast and reliable computation of statistical outputs of stochastic elliptic partial differential equations. Our method consists of three main ingredients: (1) the hybridizable discontinuous Galerkin (HDG) discretization of elliptic partial differential equations (PDEs), which allows us to obtain high-order accurate solutions of the governing PDE; (2) the reduced basis method for a new HDG discretization of the underlying PDE to enable real-time solution of the parameterized PDE in the presence of stochastic parameters; and (3) a multilevel variance reduction method that exploits the statistical correlation among the different reduced basis approximations and the high-fidelity HDG discretization to accelerate the convergence of the Monte Carlo simulations. The multilevel variance reduction method provides efficient computation of the statistical outputs by shifting most of the computational burden from the high-fidelity HDG approximation to the reduced basis approximations. Furthermore, we develop *a posteriori* error estimates for our approximations of the statistical outputs. Based on these error estimates, we propose an algorithm for optimally choosing both the dimensions of the reduced basis approximations and the sizes of Monte Carlo samples to achieve a given error tolerance. We provide numerical examples to demonstrate the performance of the proposed method.

1. Introduction

The analysis of physical systems is often carried out by mathematical modeling and numerical simulation. For a given system, the corresponding mathematical model requires certain input data, such as material properties, forcing terms, boundary conditions and geometry information. For many problems of interest, input data are not known precisely. In such cases, one may need to consider input data as random variables and represent them in probabilistic terms. Mathematical models represented by partial differential equations with random input data are known as stochastic partial differential equations (SPDEs). Uncertainty in the input data may come from different sources. It can be that the physical system has some in-

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trinsic variability, for example, uncertainty in the gust loads on an aircraft, or wind and seismic loading on civil structures. It is also possible that we are unable to effectively characterize the physical system with a mathematical model because, for instance, we may have errors in geometry, roughness of surfaces, or multiscale behavior that we are unable to capture. Therefore, there is a growing need to represent the uncertainty in the data and effectively propagate it through the mathematical model. The goal of this probabilistic approach resides in computing statistics of some observable outputs (quantities of interest), which are usually defined as functionals of the solution of the underlying SPDE.

There exist a number of different approaches to solve SPDEs and retrieve the statistics of the output. The most common approach is to use Monte Carlo (MC) methods [20,31]. Monte Carlo methods only need repeated evaluations of the output functional of the solution of the SPDEs for different instantiations of the random input. The main advantage of Monte Carlo methods is that their convergence rate is independent of the dimension of the stochastic space, namely, the number of random variables. The main caveat of these methods is their slow convergence rate, which demands a large amount of realizations to achieve accurate results. As a result, a number of techniques such as quasi Monte Carlo methods [8,48], Latin Hypercube Sampling [33,57], variance reduction methods [8] and multilevel Monte Carlo [27] have been proposed to alleviate the slow convergence rate of the standard Monte Carlo methods.

Another approach is stochastic Galerkin methods, first introduced by Ghanem et al. in [26], that generalize the theory of Wiener–Hermite polynomial chaos expansion [63] and combine it with a finite element method to model uncertainty in a SPDE. In this approach, the random variables are treated as additional dimensions of the problem and projected onto a stochastic space spanned by a set of orthogonal polynomials. The problem is then reduced to a system of deterministic equations, which couple the physical and stochastic dimensions. This methodology has proven to be very effective when solving SPDEs in a broad range of applications, such as diffusion problems and heat conduction [24,65,67], structural dynamics [25], transport in random media [23] and fluid dynamics [10,66]. The advantage of these methods is that they converge exponentially fast for a sufficiently regular solution field [2,3,18]. However, their main drawback is that their computational complexity grows combinatorially with the number of random variables and the number of expansion terms. As a consequence, they are not effective for solving problems with a large number of random variables.

A more recent approach is stochastic collocation methods (SCM), first introduced in [35] and further developed in [64]. The main idea is to compute deterministic solutions of the SPDE for certain instantiations of the random variables and then construct an interpolation function to approximate the response over the stochastic space. When the interpolation procedure is performed on tensor grids, these methods suffer from the exponential growth with the dimensionality of the stochastic space. To economize the interpolation process in large dimensions, sparse grids (Smolyak [56]) were introduced for elliptic problems [51,64], parabolic problems [49] and natural convection problems [21]. In [1], sparse grids were shown to achieve exponential convergence for problems with smooth solutions. However, like polynomial chaos expansions, sparse grids still suffer from the curse of dimensionality in the sense that the number of grid points grows rapidly with the dimension of the stochastic space. Recently, anisotropy and adaptivity on sparse grids [22,32] have been used in SCM [21,50] to mitigate the elevated cost in high dimensions.

In this paper, we present a model and variance reduction method for the fast and reliable computation of statistical outputs of stochastic elliptic partial differential equations. Our method consists of three main ingredients: (1) the hybridizable discontinuous Galerkin (HDG) discretization of elliptic partial differential equations (PDEs), which allows us to obtain high-order accurate solutions of the governing PDE; (2) a reduced basis method for the HDG discretization of the underlying PDE to enable real-time solution of the parameterized PDE in the presence of stochastic parameters; and (3) a multilevel variance reduction method that exploits the statistical correlation among the different reduced basis approximations and the high-fidelity HDG discretization to accelerate the convergence rate of the Monte Carlo simulations. The multilevel variance reduction method provides efficient computation of the statistical outputs by shifting most of the computational burden from the high-fidelity HDG approximation to the reduced basis approximations. Although the three ingredients of our approach exist in the literature, the main contribution of this paper is to put these methodologies into a unified framework that combines all of their strengths to tackle stochastic elliptic PDEs. Another important contribution of the paper is to develop *a posteriori* error bounds for the estimates of the statistical outputs and to introduce an algorithm for optimally choosing the dimensions of the reduced basis approximations and the sizes of MC samples to achieve a given error tolerance. Last but not least, we present a new HDG formulation that enables the efficient construction of reduced basis approximations for the HDG discretization of parameterized PDEs.

The HDG method was first introduced in [14] for elliptic problems, subsequently analyzed in [13,16,17], and later extended to a wide variety of PDEs [15,38–46,59]. The HDG method is particularly effective for solving elliptic PDEs because it possesses several unique features that distinguish it from other DG methods. First, it reduces the number of globally coupled unknowns to those required to represent the trace of the approximate solution on the element boundaries, thereby resulting in a smaller global system than other DG methods. Second, the method provides optimal convergence rates for both the solution and the flux. And third, its flux superconvergence properties can be exploited to devise a local postprocess that increases the convergence rate of the approximate solution by one order. These advantages are the main driver for the development of the Reduced Basis (RB) method for the HDG discretization of parameterized PDEs. While the RB method is well developed for the standard finite element discretization of parameterized PDEs [28,29,34,52,54,60,61], the RB method for the HDG approximation of parameterized PDEs has not been considered before. The HDG discretization has multiple field variables and various equivalent weak formulations, which make the application of the RB method non-straightforward.