Journal of Computational Physics 295 (2015) 379–401

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A reduced basis localized orthogonal decomposition

Assyr Abdulle [a], Patrick Henning [b,*]

[a] ANMC, Section de Mathématiques, École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland
[b] Institut für Numerische und Angewandte Mathematik, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany

A R T I C L E   I N F O

Article history:
Received 13 October 2014
Received in revised form 10 February 2015
Accepted 11 April 2015
Available online 21 April 2015

Keywords:
Finite element
Reduced basis
Parameter dependent PDE
Numerical homogenization
Multiscale method

A B S T R A C T

In this work we combine the framework of the Reduced Basis method (RB) with the framework of the Localized Orthogonal Decomposition (LOD) in order to solve parametrized elliptic multiscale problems. The idea of the LOD is to split a high dimensional Finite Element space into a low dimensional space with comparably good approximation properties and a remainder space with negligible information. The low dimensional space is spanned by locally supported basis functions associated with the node of a coarse mesh obtained by solving decoupled local problems. However, for parameter dependent multiscale problems, the local basis has to be computed repeatedly for each choice of the parameter. To overcome this issue, we propose an RB approach to compute in an "offline" stage LOD for suitable representative parameters. The online solution of the multiscale problems can then be obtained in a coarse space (thanks to the LOD decomposition) and for an arbitrary value of the parameters (thanks to a suitable "interpolation" of the selected RB). The online RB-LOD has a basis with local support and leads to sparse systems. Applications of the strategy to both linear and nonlinear problems are given.

1. Introduction

In this paper, we consider parametrized linear elliptic multiscale problems, i.e. we are interested in finding the parameter-dependent solution $u^\varepsilon(\cdot\,;\cdot)$ of an equation

$$\begin{aligned} -\nabla\cdot\big(a^\varepsilon(x;\boldsymbol{\mu})\nabla u^\varepsilon(x;\boldsymbol{\mu})\big) &= f(x;\boldsymbol{\mu}) \qquad \text{in } \Omega, \\ u^\varepsilon(x;\boldsymbol{\mu}) &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here, $\boldsymbol{\mu}=(\mu_1,\dots,\mu_P)$ denotes a parameter vector. It is an element of a multidimensional parameter set $\mathcal{D}\subset\mathbb{R}^P$, where $P\in\mathbb{N}$. The parameter-dependent coefficient matrix $a^\varepsilon(x;\boldsymbol{\mu})$ is assumed to be a *multiscale coefficient*. It exhibits a continuum of different scales, where the finest scale is very small compared to the size of computational domain Ω. In particular $a^\varepsilon(x;\boldsymbol{\mu})$ shows very fast variations that need to be resolved with an extremely fine computational grid. The order of the fines scale in our problem is characterized by the abstract quantity $0<\varepsilon\ll 1$. However, we do not need to assign a specific value to ε. Due to the requirement that all scales of $a^\varepsilon(\cdot;\boldsymbol{\mu})$ need to be resolved with a computational grid, the problem

* Corresponding author.
E-mail addresses: Assyr.Abdulle@epfl.ch (A. Abdulle), Patrick.Henning@wwu.de (P. Henning).

http://dx.doi.org/10.1016/j.jcp.2015.04.016

cannot be tackled by standard methods (such as classical finite element methods) since the computational complexity would become prohibitively large. Hence we are interested in finding a way to decrease the computational complexity and to distribute the load on several CPUs by introducing fully decoupled local subproblems. Furthermore, we want to avoid recomputing local subproblems for every new parameter $\boldsymbol{\mu}$. We are thus looking for (a small number of) representative parameters for which accurate local problems and bases are computed and that allow for fast computations for every new parameter $\boldsymbol{\mu}$.

Parameter-dependent multiscale problems can for instance arise in applications from material sciences, geophysics or hydrology. More specific examples are the prediction of global strain or elasticity properties of fiber reinforced composite materials, where the parameters can describe different constellations for the microscopic fibers that are embedded in the main material (e.g. their form or density). Another example is the flow in porous media where different permeability configurations can be parametrized. For such cases the coefficient $a^{\varepsilon}(\cdot;\boldsymbol{\mu})$ and the source term $f(\cdot;\boldsymbol{\mu})$ can both depend on a large number of parameters $\boldsymbol{\mu}$. It is therefore of strong interest to construct methods that combine the features of a multiscale method (to treat the rapid variations in the coefficients) with a reduced basis approach (to treat the dependency on a large set of parameters).

There are numerous different methods that are designed to treat the classical (parameter-free) multiscale problems (cf. [1,2,9,10,15,19,23–25,33–36,41,42,46,47] and the references therein). In this paper we focus on the localized orthogonal decomposition (LOD) introduced in [42]. To handle parameter dependency in an efficient way we will build on the reduced basis (RB) approach (cf. [26,40,48–50,53]). The reduced basis method is a model order reduction technique that we describe at the end of this section when we describe the idea of the reduced basis localized orthogonal decomposition approach (RB-LOD).

Despite the large number of results on multiscale methods and reduced basis approaches there are only few works which combine both features. In the context of periodic homogenization this was first studied by Boyaval [13,14] and extended for more general numerical homogenization problems in [3–6], where the *reduced basis finite element heterogeneous multiscale method* (RB-FE-HMM) has been introduced. The RB-FE-HMM was originally designed to reduce the computational complexity of the classical Heterogenous Multiscale Method [19] by interpreting the location of a cell problem as a parameter (which is equivalent to the dependency on the coarse variable). With that strategy, precomputed solutions from other cell problems can be used to construct reduced basis solution spaces for new cell problems. This method also generalizes to additional parameter dependencies such as in (1). A similar approach which also fits into the HMM framework was presented in [44], where the focus is on optimization problems that are constrained by a parameterized multiscale problem. A combination of the RB framework with the multiscale finite element method (MsFEM, see [34]) was proposed by Nguyen in [43], model reduction techniques for the MsFEM have also been developed in [20]. Finally, we mention the approach of the localized reduced basis multiscale method (LRBMS) proposed in [8,39] and further developed in [38,45]. The main idea of the method is to localize global solutions (that were determined for a set of parameters) to the elements of a coarse grid. The localization can be simply obtained by truncation and hence the localized solutions can be used as basis functions in a global discontinuous Galerkin approach. The Reduced Basis framework can be combined with most of the multiscale methods mentioned in the introduction. In this paper we chose the LOD because it has some attractive features compared to other approaches that also aim to solve multiscale problems without scale separation. For instance, even though an RB Multiscale Finite Element Method (RB-MsFEM, cf. [43]) is computationally less expensive, it suffers from the constrained that it requires strong structural assumptions on $a^{\varepsilon}(\cdot,\boldsymbol{\mu})$ (such as local periodicity). Efficient and reliable methods that do not suffer from such a constrained are for instance the approaches proposed by Owhadi and Zhang [46] or Babuška and Lipton [10]. The approach by Owhadi and Zhang exploits a so-called transfer property (comparable to a harmonic coordinate transformation) and requires to solve local problems in patches of sizes of order $\sqrt{H}|\log(H)|$ to guarantee an optimal linear convergence rate in H for the H^1-error. Compared to that, the LOD only requires patches with a diameter of order $H|\log(H)|$. The method of Babuška and Lipton [10] has a different structure and even smaller patches can be picked. Here optimal local approximation spaces are constructed. However, this local construction requires to incorporate the source term $f(\cdot,\boldsymbol{\mu})$ by solving additional local problems of the structure $-\nabla\cdot(a^{\varepsilon}(\cdot,\boldsymbol{\mu})\nabla v^{\varepsilon}(\cdot,\boldsymbol{\mu}))=f(\cdot,\boldsymbol{\mu})$. In order to account for this in the RB-framework, an affine decomposition of $f(\cdot,\boldsymbol{\mu})$ is required. Furthermore, the costs for the offline phase are increased. Compared to that, the LOD-approach involves local spaces that are independent of f, without suffering from a reduction of the convergence rates.

In this paper we introduce the reduced basis local orthogonal decomposition (RB-LOD). We briefly summarize the main ideas. Consider a coarse triangulation $\mathcal{T}_H$ and a corresponding set of coarse nodes $\mathcal{N}_H$. For any fixed (i.e. parameter independent) coefficient a^{ε} the LOD is designed to construct a set of (locally supported) multiscale basis functions Φ_z^{MS} (each of them associated with a single coarse node $z\in\mathcal{N}_H$) so that the discrete space that is spanned by these basis functions yields the classical convergence rates in H. The functions Φ_z^{MS} are obtained from the solution of a local finite element problem (in a local space that resolves the microstructure). The coarse triangulation $\mathcal{T}_H$ does not need to resolve the microstructure and can hence be low dimensional. However if the coefficient $a^{\varepsilon}(\cdot;\boldsymbol{\mu})$ is parameter-dependent then $\Phi_z^{\mathrm{MS}}(\boldsymbol{\mu})$ is parameter-dependent as well and needs to be recomputed again for any new parameter. To overcome this drawback we apply the reduced basis method together with a Greedy search algorithm to identify a set of parameters for which we compute Φ_z^{MS}. These solutions can be used to construct affine (reduced basis) spaces V_z^{RB}, for each node $z\in\mathcal{N}_H$.

The computation of the spaces V_z^{RB} takes place in an offline phase (i.e. it is a preprocessing step). The functions in V_z^{RB} are only locally supported in a small patch around the node z. Once constructed, these reduced basis (multiscale) spaces