Journal of Computational Physics 295 (2015) 402–419

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A split control variate scheme for PIC simulations with collisions

Eric Sonnendrücker [a,c,*], Abigail Wacher [a], Roman Hatzky [a], Ralf Kleiber [b]

[a] Max Planck Institute for Plasma Physics, Boltzmannstr. 2, 85748 Garching, Germany
[b] Max Planck Institute for Plasma Physics, Wendelsteinstr. 1, 17491 Greifswald, Germany
[c] Mathematics Center, TU Munich, Boltzmannstr. 3, 85747 Garching, Germany

ARTICLE INFO

Article history:
Received 9 May 2014
Received in revised form 27 February 2015
Accepted 6 April 2015
Available online 22 April 2015

Keywords:
Vlasov–Poisson system
Fokker–Planck
Collisions
Particle in cell
PIC
Monte Carlo
Control variate

ABSTRACT

When the distribution function of plasma particles stays close to some analytically known function, statistical noise inherent to Monte Carlo simulations can be greatly reduced by introducing this function as a control variate in the computation of the velocity moments. Such a method, even though it can be naturally applied to nonlinear simulations, has originally emerged from linearised simulations and is usually called the δf particle-in-cell (PIC) method. In the past, the method has been extended to also handle collisions. This resulted in a two weight scheme which is known to produce a pronounced weight growth problem which rapidly makes it inefficient as a control variate method for variance reduction. In this work we analyse the weight growth problem within a simple example, which allows us to overcome its pathological behaviour. We also introduce a new split algorithm based on switching the control variate for PIC simulations with collisions. A key element of our algorithm is a new weight smoothing operator which enables us to obtain a significant noise reduction both in the presence of collisions and in the deep nonlinear phase of PIC simulations.

1. Introduction

The particle in cell (PIC) method is a well established method for the numerical simulation of collisional or collisionless kinetic plasmas [4]. Its main drawback compared to grid based methods is its numerical noise that is slowly decreasing (as $1/\sqrt{N}$) with the number of particles. In some situations, like in gyrokinetic simulations of magnetic fusion plasmas, using a standard PIC method requires either a huge number of particles or generates noise levels above the signal level. This could be remedied by the so-called δf approach [10] which allows to decrease the noise by orders of magnitude by simulating with particles only the deviation from equilibrium. This method involving linearisation in its early stages can be made completely nonlinear. Aydemir [3] pointed out that this method could be interpreted as a control variate method classically used in Monte Carlo simulations. This method has been proven to be very efficient and useful for collisionless gyrokinetic simulations. However, it has been very difficult to efficiently handle collisions with this method. The two weight scheme introduced in [5] has been prone to an artificial weight growth [8] that is problematic for long time simulations as it can increase the variance and thus also the noise drastically. Even though the coarse graining method [7] allows to alleviate the problem it is still not completely satisfying.

* Corresponding author at: Max Planck Institute for Plasma Physics, Boltzmannstr. 2, 85748 Garching, Germany.
E-mail address: eric.sonnendruecker@ipp.mpg.de (E. Sonnendrücker).

http://dx.doi.org/10.1016/j.jcp.2015.04.004
0021-9991/

In this paper we motivate a new method for PIC simulations with collisions using a control variate approach. One of its features is the idea of switching the control variate between the advection and the collision step, which has been first introduced by Vernay et al. [15,16]. We proceed by analysing the error introduced by the two weight δf particle in cell method for solving the collisional Vlasov equation, and observe that the pathological weight growth can be eliminated by using a well suited control variate during the collision step, an equilibrium function of the collision operator, which can be usually computed in practice. This brings us to the formulation of our algorithm based on a splitting between the advection step and the collision step, using at every collision step the actual equilibrium of the collision operator as a control variate. Any other function (close enough) can be used as a control variate during the advection phase, the algorithm then involving a switching of control variates between the advection and collision steps. In practice this method only involves the evolution of one weight with no time integration during the collision step. This completely suppresses the weight growth during the collision step, however a slight weight growth is still observed due to the advective and rescaling stages.

Moreover, the collision step should not only be free of weight growth but also smooth the weights so that the collisions actually bring back the distribution to equilibrium and thus diminish the weights at that step. This could be realised by a kernel density estimate which is akin to coarse graining. Instead, we propose a simpler procedure which is grid less and only involves pairs of close particles, thus making it completely local, which is a big asset for large parallel computations. We call this: neighbour smoothing procedure. We will show that adding this step to our algorithm allows to have an appropriate relaxation of the distribution function towards the equilibrium with a corresponding reduction of the variance of the weights and results in better statistics.

The article is organised as follows. First we introduce an abstract model for a collisional phase space transport. In our examples it will represent the Vlasov–Fokker–Planck equation, but it could also be the gyrokinetic equations with collisions. This model is split into two parts, the first one defining an advection along some characteristics and the second a collision operator that admits a known steady state. Then both parts are considered individually, the collision operator being cast into the framework of stochastic differential equations (SDE). A split algorithm for the solution of the initial problem is then introduced. After that, a new neighbour smoothing procedure is introduced to be used in addition with the previous algorithm to ensure a pointwise convergence of the distribution function to the equilibrium distribution in the case of strong collisions. A section is then devoted to the analysis of the weight growth problem that plagues standard collisional two weights PIC simulations. And finally we validate the new methods introduced in the paper on relevant test cases.

2. An abstract collisional Vlasov equation

Denoting by $\mathbf{z} = (x, v)$ the phase space variables we consider a collisional Vlasov equation with a known source term of the form

$$\frac{\partial f}{\partial t} + \mathbf{A} \cdot \nabla f = -\nabla \cdot (\mathbf{A}_C f) + \frac{1}{2}\nu^2 \Delta_v f + \tilde{\mathcal{S}}(f), \tag{1}$$

where $\nabla := (\partial_x, \partial_v)$ acts on the phase space and the diffusion operator Δ_v acts in velocity space only. $\tilde{\mathcal{S}}(f)$ is a general source term. This abstract model contains the Vlasov–Fokker–Planck model as well as the gyrokinetic equations with a linear collision operator.

The linearity of the collision operator assumed for simplicity in this work is not required for our algorithm. It would also be possible to use the nonlinear Landau collision operator expressed with Rosenbluth potentials.

In our particle in cell scheme we are going to treat independently the advection and collision term using a time splitting technique, the advection part being

$$\frac{\partial f}{\partial t} + \mathbf{A} \cdot \nabla f = \tilde{\mathcal{S}}(f), \tag{2}$$

and the collision part

$$\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{A}_C f) - \frac{1}{2}\nu^2 \Delta_v f = 0. \tag{3}$$

This splitting is somewhat arbitrary since for a given equation we have some freedom in what to put into $\mathbf{A}$ and what into $\mathbf{A}_C$. However a strong requirement for our method to work is that the collision part Eq. (3) admits an analytically known non-trivial steady-state, which is typically a local Maxwellian.

3. The δf method in the pure advection case

The standard variance reduction technique used in the PIC method, is the so-called δf method, where a control variate associated with a known equilibrium distribution close to the computed distribution is used [8].

Let us introduce an arbitrary function $f^0(t, \mathbf{z})$ that is analytically known. Then define, by explicit computation

$$\mathcal{S}^0 = -\frac{\partial f^0}{\partial t} - \mathbf{A}[f] \cdot \nabla f^0. \tag{4}$$