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Entropy-bounded discontinuous Galerkin scheme for Euler equations

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ABSTRACT

An entropy-bounded Discontinuous Galerkin (EBDG) scheme is proposed in which the solution is regularized by constraining the entropy. The resulting scheme is able to stabilize the solution in the vicinity of discontinuities and retains the optimal accuracy for smooth solutions. The properties of the limiting operator according to the entropy-minimum principle are proofed, and an optimal CFL-criterion is derived. We provide a rigorous description for locally imposing entropy constraints to capture multiple discontinuities. Significant advantages of the EBDG-scheme are the general applicability to arbitrary high-order elements and its simple implementation for multi-dimensional configurations. Numerical tests confirm the properties of the scheme, and particular focus is attributed to the robustness in treating discontinuities on arbitrary meshes.

1. Introduction

The stabilization of solutions near flow-field discontinuities remains an open problem to the discontinuous Galerkin (DG) community. Considerable progress has been made on the development of limiters for two-dimensional quadrilateral and triangular elements. These limiters can be categorized into three classes. Methods that limit the solution using information about the slope along certain spatial directions [1,2] fall in the first class. The second class of limiters extends this idea by limiting based on the moments of the solution [3,4], and schemes in which the DG-solution is projected onto a WENO [5–7] or Hermit WENO (HWENO) [8] representation fall in the last category. Although these limiters show promising results for canonical test cases on regular elements and structured mesh partitions, the following two issues related to practical applications have not been clearly answered:

- How can discontinuous solutions be regularized on multi-dimensional curved high-order elements?
- How can non-physical solutions that are triggered by strong discontinuities and geometric singularities be avoided?

The present work attempts to simultaneously address both of these questions.

Recently, positivity-preserving DG-schemes have been developed for the treatment of flow-field discontinuities, and relevant contributions are by Zhang and Shu [9–11]. The positivity preserving method provides a robust framework with provable L_1-stability, preventing the appearance of negative pressure and density. Resulting algorithmic modifications are minimal, and these schemes have been used in simulations of detonation systems with complex reaction chemistry [12,13].

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Motivated by these attractive properties, the present work aims at developing an algorithm that avoids non-physical solutions on arbitrary elements and multi-dimensional spatial representations. The resulting scheme that will be developed in this work has the following properties: First, by invoking the entropy principle, solutions are constrained by a local entropy bound. Second, a general implementation on arbitrary elements is proposed without restriction to a specific quadrature rule. Third, the entropy constraint is imposed on the solutions through few algebraic operations, thereby avoiding the computationally expensive inversion of a nonlinear system. Fourth, a method for the evaluation of an optimal CFL-criterion is derived, which is applicable to general polynomial orders and arbitrary element types.

The remainder of this paper has the following structure. The governing equations and the discretization are summarized in the next two sections. The entropy-bounded DG (EBDG) formulation is presented in Section 4, and the derivation of the CFL-constraint and the limiting operator are presented. This analysis is performed by considering a one-dimensional setting, and the generalization to multi-dimensional and arbitrary elements is presented in Section 5. Section 6 is concerned with the evaluation of the entropy-bounded DG-scheme, and a detailed description of the algorithmic implementation is given in Section 7. The EBDG-method is demonstrated by considering several test cases, and the accuracy and stability are examined in Section 8. The paper finishes with conclusions.

2. Governing equations

We consider a system of conservation equations,

$$\frac{\partial \mathsf{U}}{\partial t} + \nabla \cdot \mathsf{F} = 0 \qquad \text{in } \Omega\,, \tag{1}$$

where the solution variable $\mathsf{U} : \mathbb{R}^{N_d} \times \mathbb{R} \to \mathbb{R}^{N_v}$ and the flux term $\mathsf{F} : \mathbb{R}^{N_v} \to \mathbb{R}^{N_v \times N_d}$. Here, N_d denotes the spatial dimension and N_v is the dimension of the solution vector. For the Euler equations, U and F take the form:

$$\mathsf{U}(x,t) = (\rho, \rho u, \rho e)^T\,, \tag{2a}$$

$$\mathsf{F}(\mathsf{U}) = (\rho u, \rho u \otimes u + p\mathrm{I}, u(\rho e + p))^T\,, \tag{2b}$$

where t is the time, $x \in \mathbb{R}^{N_d}$ is the spatial coordinate vector, ρ is the density, $u \in \mathbb{R}^{N_d}$ is the velocity vector, e is the specific total energy, and p is the pressure. Eq. (1) is closed with the ideal gas law:

$$p = (\gamma - 1)\left(\rho e - \frac{\rho |u|^2}{2}\right)\,, \tag{3}$$

in which γ is the ratio of specific heats, which, for the present work, is set to a constant value of $\gamma = 1.4$. Here and in the following, we use $|\cdot|$ to represent the Euclidean norm. With this, we define the local maximum characteristic speed as:

$$\nu = |u| + c \qquad \text{with} \qquad c = \sqrt{\frac{\gamma p}{\rho}}\,, \tag{4}$$

where c is the speed of sound.

Because of the presence of discontinuities in the solution of Eq. (1), we seek a weak solution that satisfies physical principles. This is the so-called entropy solution. By introducing $\mathcal{U}$ as a convex function of U with $\mathcal{U} : \mathbb{R}^{N_v} \to \mathbb{R}$, Lax [14] showed that the entropy solution of Eq. (1) satisfies the following inequality:

$$\frac{\partial \mathcal{U}}{\partial t} + \nabla \cdot \mathcal{F} \le 0\,, \tag{5}$$

where $\mathcal{F} : \mathbb{R}^{N_v} \to \mathbb{R}^{N_d}$ is the corresponding flux of $\mathcal{U}$. The consistency condition between Eqs. (1) and (5) requires [14]:

$$\left(\frac{\partial \mathcal{U}}{\partial \mathsf{U}}\right)^T \frac{\partial \mathsf{F}}{\partial \mathsf{U}} = \frac{\partial \mathcal{F}}{\partial \mathsf{U}}\,. \tag{6}$$

The weak solution of Eq. (1) that satisfies this additional condition for the entropy–entropy flux pair $(\mathcal{U}, \mathcal{F})$ is called an entropy solution. With this definition, Eq. (5) is commonly called entropy inequality or entropy condition, and $\mathcal{U}$ is called entropy function. A familiar example for gas-dynamic applications is to relate $\mathcal{U}$ to the physical entropy s with:

$$s = \ln(p) - \gamma \ln(\rho) + s_0\,, \tag{7}$$

where s_0 is the reference entropy. The corresponding definitions for the entropy function and entropy flux in the context of the Euler system are [15]:

$$(\mathcal{U}, \mathcal{F}) = (-\rho s, -\rho s u)\,. \tag{8}$$

Note that Eq. (7) directly provides a constraint on the positivity of pressure p and density ρ.