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Numerically accurate computation of the conditional trajectories of the topological invariants in turbulent flows

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ABSTRACT

The computation of the topological invariants of the velocity gradient tensor and of their conditional mean trajectories in incompressible turbulent flows is revisited. It is argued that probability conservation requires that the conditional mean trajectories should be closed when a statistically stationary wall-bounded or periodic domain is considered, and this is confirmed numerically for a turbulent channel. It is argued that previous reports of inward spiraling of the conditional trajectories are either due to incomplete statistics in inhomogeneous flows or to numerical errors.

1. Introduction

Many fundamental properties of turbulent flows are encoded in the gradients of the velocity, which provide important information regarding the local behavior of the flow. The invariants of the velocity gradient tensor for incompressible flows, R and Q, were first introduced in [1] and have proven to be a useful tool to analyze turbulent flows characterized by a wide spectrum of scales, e.g. [1–9]. They quantify the relative strength of enstrophy production and strain self-amplification, and of local enstrophy density and strain density respectively. Moving locally with a fluid particle, the velocity gradient tensor determines the linear approximation to the local velocity field surrounding the observer. In that frame, invariants can also be used to classify the local flow topology, as shown in [1].

R and Q are challenging quantities to obtain from both the numerical and experimental point of view, since they involve products of the velocity gradients and it is crucial to represent accurately the high wavenumbers. Nevertheless, they have been measured experimentally and computed numerically numerous times. Results from different turbulent flows showed that the joint probability density function of R and Q has a very particular skewed 'tear drop' shape, e.g. [10,11,2,3]. That is, there is an increased probability of points where $R > 0$ and $Q < 0$ along the so-called Vieillefosse tail. Such a signature turns out to be a quite universal feature persistent in many different turbulent flows, including mixing layers [10], channel flows [12], boundary layers [13,7], homogeneous isotropic turbulence [14,11], etc.

References [14] and [11] introduced and studied the conditional mean trajectories of the invariants (henceforth, CMTs) in direct numerical simulations of homogeneous isotropic turbulence at low Reynolds numbers. This involved the calculation of the mean temporal rate of change of the invariants conditioned on the values of the invariants themselves, and gives a vector field in the R–Q plane. The resulting conditional vector field can be integrated to produce trajectories within the

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space of the pair of the invariants. The computation of this vector field is even more challenging than the one of R and Q since it requires calculating their material derivative.

The analysis of the CMTs yields information about the dynamics of the small scales of turbulence. Previous results suggested a cyclic and approximately periodic orbit with a mean clockwise evolution in the R–Q plane and trajectories spiraling towards $(R, Q) = (0, 0)$. Some authors have conjectured about the spurious nature of the spiraling of the CMTs towards the origin [14]. However, others have argued that this effect may have a physical significance related to the statistical tendency of the flow to form shear layers [15] and the question remains open. The time-scale associated with the CMTs is also considered representative of the cycle involved in the dynamics of turbulence and has been computed in [14,11,6,15] and [7] among others.

The dynamical evolution of the velocity gradient tensor, and hence, the accurate computation of the CMTs, is also fundamental for the development of statistical models (see [16] and references therein). Due to its simplicity, one of the first models was the Restricted Euler Model [17,18], where the influence of the pressure and viscosity is neglected. Its solution was later obtained and analyzed in [19] but it showed neither closed nor spiraling CMTs and its oversimplified dynamics resulted in a qualitatively different picture from that obtained in simulated and experimental results. Later attempts focused on more sophisticated models which reproduced more realistic CMTs without determining whether the trajectories should or should not be closed [16]. It is clear that these models should be conceptually different depending on the nature of the CMTs, and from there derives the importance of its study.

In this paper, we study the numerical requirements for the accurate computation of the invariants of the velocity gradient tensor in turbulent channel flows and its implications in the CMTs. The effects of the normalization of the invariants and of the inhomogeneity of the flow are addressed too. It is also analyzed which are the necessary conditions for the CMTs to form closed trajectories.

The paper is organized as follows. The invariants of the velocity gradient tensor and the conditions to form closed CMTs are discussed in Section 2. The numerical experiments and methods are presented in Section 3. Results are offered in Section 4, which is divided in three subsections. Section 4.1 is devoted to the numerical analysis of the computation of the invariants and its material derivatives, Section 4.2 to the effect of the normalization, and Section 4.3 to the consequences of conditioning the statistics on sub-domains in inhomogeneous flows. Finally, we close with the conclusions in Section 5.

2. Invariants of the velocity gradient tensor and probability conservation

The invariants of the velocity gradient tensor for an incompressible flow, Q and R, are

$$Q = \frac{1}{4}(\omega_i\omega_i - 2s_{ij}s_{ij}), \tag{1}$$

$$R = -\frac{1}{3}s_{ij}s_{jk}s_{ki} - \frac{1}{4}\omega_i\omega_j s_{ij}, \tag{2}$$

where summation over repeated indices is implied, ω_i are the components of the vorticity vector and s_{ij} of the rate-of-strain tensor.

The invariants R and Q defined by relations (1) and (2) may be interpreted in two ways. From a physical point of view, Q measures the relative importance of enstrophy and strain densities. Enstrophy dominates over strain for positive and large values of Q, and strain does for negative and large ones. The meaning of R depends on the value of Q. For $Q > 0$ and large, $R < 0$ represents vortex stretching and $R > 0$ vortex compression. For $Q < 0$ and large, R is dominated by the strain self-amplification. The second interpretation is topological, R and Q characterize the local motion of the fluid particles for an observer traveling with the fluid. The lines $D = 27/4R^2 + Q^3 = 0$ and $R = 0$ divide the R–Q plane in four regions (see Fig. 2(a)). The trajectories of the fluid particles are then classified according to critical point terminology [1], as stable focus/stretching (upper left-hand region), unstable focus/compressing (upper right-hand region), stable node/saddle/saddle (lower left-hand region) and unstable node/saddle/saddle (lower right-hand region).

The conditional mean trajectories or CMTs [14] aim to study the Lagrangian temporal evolution of the invariants. The method relies on calculating the averaged time rates of change of the invariants for the fluid particles, $\mathrm{D}R/\mathrm{D}t$ and $\mathrm{D}Q/\mathrm{D}t$, conditioned on the values of R and Q. Note that $\mathrm{D}/\mathrm{D}t$ stands for material derivative. These quantities can be thought of as the components of a conditionally averaged vector field in the R–Q plane,

$$\boldsymbol{v} = \left\langle \left(\frac{\mathrm{D}R}{\mathrm{D}t}, \frac{\mathrm{D}Q}{\mathrm{D}t} \right) \right\rangle_{R,Q}, \tag{3}$$

where $\langle\cdot\rangle_{R,Q}$ denotes conditional average at point (R, Q). From $\boldsymbol{v}$, any chosen initial condition can be integrated resulting in the aforementioned CMTs.

If we define $J(R, Q)$ as the joint probability density function of R and Q, the equation for the conservation of probability states that

$$\frac{\partial J}{\partial t} + \begin{pmatrix} \frac{\partial}{\partial R} \\ \frac{\partial}{\partial Q} \end{pmatrix} \cdot \boldsymbol{\mathcal{W}} = 0, \tag{4}$$