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Fully discrete energy stable high order finite difference methods for hyperbolic problems in deforming domains

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ABSTRACT

A time-dependent coordinate transformation of a constant coefficient hyperbolic system of equations which results in a variable coefficient system of equations is considered. By applying the energy method, well-posed boundary conditions for the continuous problem are derived. Summation-by-Parts (SBP) operators for the space and time discretization, together with a weak imposition of boundary and initial conditions using Simultaneously Approximation Terms (SATs) lead to a provable fully-discrete energy-stable conservative finite difference scheme.

We show how to construct a time-dependent SAT formulation that automatically imposes boundary conditions, when and where they are required. We also prove that a uniform flow field is preserved, i.e. the Numerical Geometric Conservation Law (NGCL) holds automatically by using SBP-SAT in time and space. The developed technique is illustrated by considering an application using the linearized Euler equations: the sound generated by moving boundaries. Numerical calculations corroborate the stability and accuracy of the new fully discrete approximations.

1. Introduction

High order SBP operators together with weak implementation of boundary conditions by SATs, can efficiently and reliably handle large problems on structured grids for reasonably smooth geometries [1–7]. The main reason to use weak boundary procedures together with SBP operators and the energy method is the fact that with this combination, provable stable schemes can be constructed. For comprehensive reviews of the SBP-SAT schemes, see [8,9].

The developments described above have so far dealt mostly with steady problems while computing flow-fields around moving and deforming objects involves time-dependent meshes [10–12]. We have previously treated the problems with steady coordinate transformations [11,5,6]. In this paper we take the next step, which is the treatment of time-dependent transformations in combination with SBP-SAT schemes. To guarantee stability of the fully discrete approximation we employ the recently developed SBP-SAT technique in time [13,14].

The hyperbolic constant coefficient system that we consider, represents wave propagation problems governed by for example the elastic wave equation [15,6], Maxwell's equations [16,17,4] and the linearized Euler equations [18–20].

The rest of this paper proceeds as follows. In Section 2, we analyze the continuous problem which undergoes a transformation from a deforming domain into a fixed domain, and derive characteristic boundary conditions which lead to a strongly

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well-posed problem. Section 3 deals with the discrete problem where we guarantee stability, conservation and the validity of the NGCL. In Section 4, numerical examples which corroborate the previous theoretical development and confirm the accuracy and stability of the scheme are considered. An application where sound is generated and propagated by a moving boundary is also studied. Finally we draw conclusions in Section 5.

2. The continuous problem

Consider the following constant coefficient system,

$$V_t + (\hat{A}V)_x + (\hat{B}V)_y = 0, \quad (x,y) \in \Phi(t), \quad t \in [0,T], \tag{1}$$

where the spatial domain Φ is time-dependent. We assume for simplicity that the constant matrices $\hat{A}$ and $\hat{B}$ are symmetric and of size l. If the original problem is not symmetric, we symmetrize it by the procedure in [18].

A time-dependent transformation from the Cartesian coordinates into curvilinear coordinates, which results in a fixed spatial domain, is

$$\begin{aligned} &x = x(\tau,\xi,\eta), \quad y = y(\tau,\xi,\eta), \quad t = \tau, \\ &\xi = \xi(t,x,y), \quad \eta = \eta(t,x,y), \quad \tau = t. \end{aligned} \tag{2}$$

The chain-rule is employed to interpret the system (1) in terms of the curvilinear coordinates as

$$V_\tau + (\xi_t I + \xi_x \hat{A} + \xi_y \hat{B})V_\xi + (\eta_t I + \eta_x \hat{A} + \eta_y \hat{B})V_\eta = 0, \tag{3}$$

where $0 \le \xi \le 1,\ 0 \le \eta \le 1,\ 0 \le \tau \le T$. The Jacobian matrix of the transformation is

$$[J] = \begin{pmatrix} x_\xi & y_\xi & 0 \\ x_\eta & y_\eta & 0 \\ x_\tau & y_\tau & 1 \end{pmatrix}, \tag{4}$$

where $(V_\xi, V_\eta, V_\tau)^T = [J](V_x, V_y, V_t)^T$. The relation between $[J]$, and its inverse, which transforms the derivatives back to the Cartesian coordinates leads to the metric relations

$$\begin{aligned} &J\xi_t = x_\eta y_\tau - x_\tau y_\eta, \quad J\xi_x = y_\eta, \quad J\xi_y = -x_\eta \\ &J\eta_t = y_\xi x_\tau - x_\xi y_\tau, \quad J\eta_x = -y_\xi, \quad J\eta_y = x_\xi, \end{aligned} \tag{5}$$

in which $J = x_\xi y_\eta - x_\eta y_\xi > 0$ is the determinant of $[J]$.

By multiplying (3) with J and using (5), we replace the coefficients in terms of derivatives of the curvilinear coordinates. Eq. (3) can be rewritten as

$$\begin{aligned} &(JV)_\tau + [(J\xi_t I + J\xi_x \hat{A} + J\xi_y \hat{B})V]_\xi + [(J\eta_t I + J\eta_x \hat{A} + J\eta_y \hat{B})V]_\eta \\ &\quad = [J_\tau + (J\xi_t)_\xi + (J\eta_t)_\eta]V + [(J\xi_x)_\xi + (J\eta_x)_\eta]\hat{A}V + [(J\xi_y)_\xi + (J\eta_y)_\eta]\hat{B}V, \end{aligned} \tag{6}$$

where I denotes the identity matrix of size l. All non-singular coordinate transformations fulfill the Geometric Conservation Law (GCL) [10,21,22], which is summarized as

$$\begin{aligned} J_\tau + (J\xi_t)_\xi + (J\eta_t)_\eta &= 0, \\ (J\xi_x)_\xi + (J\eta_x)_\eta &= 0, \\ (J\xi_y)_\xi + (J\eta_y)_\eta &= 0. \end{aligned} \tag{7}$$

The right hand side of (6) is identically zero, due to (7), which results in the conservative form of the system.

The final problem in the presence of initial and boundary conditions that we will consider in this paper is

$$\begin{aligned} (JV)_\tau + (AV)_\xi + (BV)_\eta &= 0, & (\xi,\eta) &\in \Omega, & \tau &\in [0,T], \\ LV &= g(\tau,\xi,\eta), & (\xi,\eta) &\in \delta\Omega, & \tau &\in [0,T], \\ V &= f(\xi,\eta), & (\xi,\eta) &\in \Omega, & \tau &= 0, \end{aligned} \tag{8}$$

where

$$A = J\xi_t I + J\xi_x \hat{A} + J\xi_y \hat{B}, \quad B = J\eta_t I + J\eta_x \hat{A} + J\eta_y \hat{B}, \tag{9}$$

and $\Omega = [0,1] \times [0,1]$. In (8), L is the boundary operator, g is the boundary data and f is the initial data.