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High order cell-centered Lagrangian-type finite volume schemes with time-accurate local time stepping on unstructured triangular meshes

Walter Boscheri, Michael Dumbser*, Olindo Zanotti

Laboratory of Applied Mathematics, Department of Civil, Environmental and Mechanical Engineering, University of Trento, Via Mesiano 77, I-38123 Trento, Italy

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ABSTRACT

We present a novel cell-centered direct Arbitrary-Lagrangian–Eulerian (ALE) finite volume scheme on unstructured triangular meshes that is high order accurate in space and time and that also allows for *time-accurate local time stepping* (LTS). It extends our previous investigations on high order Lagrangian finite volume schemes with LTS carried out in [46] in one space dimension. The new scheme uses the following basic ingredients: a high order WENO reconstruction in space on unstructured meshes, an element-local high-order accurate space–time Galerkin predictor that performs the time evolution of the reconstructed polynomials within each element, the computation of numerical ALE fluxes at the moving element interfaces through approximate Riemann solvers, and a one-step finite volume scheme for the time update which is directly based on the integral form of the conservation equations in space–time. The inclusion of the LTS algorithm requires a number of crucial extensions, such as a proper scheduling criterion for the time update of each element and for each node; a virtual projection of the elements contained in the reconstruction stencils of the element that has to perform the WENO reconstruction; and the proper computation of the fluxes through the space–time boundary surfaces that will inevitably contain hanging nodes in time due to the LTS algorithm.

We have validated our new unstructured Lagrangian LTS approach over a wide sample of test cases solving the Euler equations of compressible gas dynamics in two space dimensions, including shock tube problems, cylindrical explosion problems, as well as specific tests typically adopted in Lagrangian calculations, such as the Kidder, the Saltzman and the Sedov problem. When compared to the traditional global time stepping (GTS) method, the newly proposed LTS algorithm allows to reduce the number of element updates in a given simulation by a factor that may depend on the complexity of the dynamics, but which can be as large as ∼4.7. Finally, we have also shown the improvement in terms of computational efficiency in a representative test for the special relativistic magnetohydrodynamics (RMHD) equations.

* Corresponding author.
E-mail addresses: walter.boscheri@unitn.it (W. Boscheri), michael.dumbser@unitn.it (M. Dumbser), olindo.zanotti@unitn.it (O. Zanotti).

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1. Introduction

In the last few years there has been a renewed interest in the development of novel accurate and robust cell-centered Lagrangian finite volume schemes for hydrodynamics. Since in the Lagrangian framework the computational mesh moves with the local fluid velocity, such schemes are regarded as the first choice in all problems presenting moving material interfaces appearing in compressible multi-phase and multi-material flows, such as in the numerical simulation of inertial confinement fusion (ICF). The vast majority of modern Lagrangian schemes adopt a *cell-centered* finite-volume approach, see for example [21,29,99,86,90–92,89], where all flow variables are defined as cell-averaged quantities inside a control volume. However, also *staggered* Lagrangian schemes are possible, see e.g. [84], where the velocity is defined at the cell interfaces, while the other flow variables are still defined at the cell centers.

In [95,21] Godunov-type finite volume schemes have been presented for Lagrangian hydrodynamics, while in [33,34] the governing equations have been coupled with the equations for the evolution of the geometry and the resulting weakly hyperbolic system has been solved using a node-based finite volume solver. Unstructured multidimensional meshes have been considered by Maire in [87,89,88], who developed up to second order accurate cell-centered Lagrangian schemes, where the time derivatives of the fluxes have been computed with a node-centered solver. This approach may be regarded as a multi-dimensional Lagrangian extension of the Generalized Riemann problem methodology used for example in the ADER approach of Titarev and Toro [105,107] in the Eulerian context. Arbitrary-Lagrangian–Eulerian (ALE) methods based on remeshing and remapping have also been investigated very recently for single and multi-material flows in [63,112,19, 100].

In [27,81] Cheng and Shu presented the first better than second order accurate Lagrangian schemes for hydrodynamics on structured meshes, where the use of a high order Essentially Non-Oscillatory (ENO) reconstruction operator yielded high order of accuracy in space, while high order of accuracy in time was guaranteed using either a Runge–Kutta or a Lax–Wendroff-type time stepping. Arbitrary high order accurate cell-centered Lagrangian-type finite volume schemes for conservative and non-conservative hyperbolic PDEs on moving unstructured triangular and tetrahedral meshes have been considered for the first time by Boscheri et al. in a very recent series of papers [15,48,17,14,16]. A new class of *meshless* Lagrangian particle methods based on a high order accurate moving least-squares WENO reconstruction has been forwarded in [3].

High order accurate Lagrangian algorithms using the classical continuous finite element method (FEM) can be found, for example, in the work of Scovazzi et al. [97,101] and Dobrev et al. [37–39], while Lagrangian discontinuous Galerkin finite elements have been recently proposed by Vilar et al. and Yu et al. in [66,64,65,80]. Arbitrary-Lagrangian–Eulerian DG schemes have been developed and applied, for example, in [26,60].

Almost all of the above mentioned algorithms use an explicit *global* time stepping scheme in which the timestep is computed under a classical *global* CFL stability condition, so that the timestep is essentially determined by the smallest control volume appearing in the mesh. In Lagrangian hydrodynamics, where the mesh follows as closely as possible the local fluid motion, very severe deformations and distortions may occur in the computational cells, especially at shocks and shear waves. As a consequence, the computational efficiency of the algorithm drastically decreases, because the smallest timestep imposed by the most deformed control volumes dictates the timestep for the entire computational grid, including those elements which are much bigger or which lie in a zone where the fluid is moving uniformly. In the Eulerian framework such a problem can be partially avoided controlling the mesh quality *a priori* and designing a high quality mesh at the pre-processing level, since the grid will not change anymore during the simulation. Of course, the CFL condition can be circumvented by using implicit or semi-implicit schemes, see for example [23,25,24,40–42], but this approach does not yet seem to be very popular in the context of cell-centered Lagrangian-type finite volume methods. An alternative to overcome the global CFL condition consists in the development of numerical schemes that allow for time-accurate *local* time stepping (LTS), where each element has to obey only a less restrictive *local* CFL stability condition, hence using its own optimal local timestep. Therefore, many efforts have been devoted to the construction of high order accurate Eulerian schemes with time-accurate LTS, developing either discontinuous Galerkin finite element methods [62,53,104,82,67,79,61] or high order accurate finite volume schemes with LTS [12,11,111,22,5,4,20,61,58,49]. The finite volume schemes with LTS adopt mainly classical adaptive mesh refinement (AMR) techniques in space and time or block-clustered local time stepping algorithms. In [70,69] also high order accurate Runge–Kutta time integrators with local time stepping (so-called multi-rate integrators) can be found. To the knowledge of the authors, the first high order accurate *Lagrangian* algorithm with *time accurate local time stepping* on moving grids has been proposed only very recently in [46], where the equations of hydrodynamics and of classical magnetohydrodynamics (MHD) have been solved in one spatial dimension. In the present paper we extend the algorithm presented in [46] to *moving unstructured triangular meshes*.

The rest of the paper is structured as follows: in Section 2 the numerical scheme is described, including the details of the local time stepping algorithm on moving unstructured meshes, while numerical convergence studies as well as some classical numerical test problems for hydrodynamics are presented in Section 3. We conclude the paper giving an outlook to future research and developments in Section 4.