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On the existence of nonoscillatory phase functions for second order ordinary differential equations in the high-frequency regime

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ABSTRACT

We observe that solutions of a large class of highly oscillatory second order linear ordinary differential equations can be approximated using nonoscillatory phase functions. In addition, we describe numerical experiments which illustrate several implications of this fact. For example, that many special functions of great interest — such as the Bessel functions J_ν and Y_ν — can be evaluated accurately using a number of operations which is $O(1)$ in the order ν. The present paper is devoted to the development of an analytical apparatus. Numerical aspects of this work will be reported at a later date.

1. Introduction

Given a differential equation

$$y''(t) + \lambda^2 q(t) y(t) = 0 \quad \text{for all } 0 \leqslant t \leqslant 1, \tag{1}$$

where λ is a real number and $q : [0,1] \to \mathbb{R}$ is smooth and strictly positive, a sufficiently smooth $\alpha : [0,1] \to \mathbb{R}$ is a phase function for (1) if the pair of functions u, v defined by the formulas

$$u(t) = \frac{\cos(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{2}$$

and

$$v(t) = \frac{\sin(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{3}$$

form a basis in the space of solutions of (1). Phase functions have been extensively studied: they were first introduced in [9], play a key role in the theory of global transformations of ordinary differential equations [3,10], and are an important element in the theory of special functions [16,6,11,1].

Despite this long history, a useful property of phase functions appears to have been overlooked. Specifically, that when the function q is nonoscillatory, solutions of Eq. (1) can be accurately represented using a nonoscillatory phase function.

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This is somewhat surprising since α is a phase function for (1) if and only if it satisfies the third order nonlinear ordinary differential equation

$$\left(\alpha'(t)\right)^2 = \lambda^2 q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 \quad \text{for all} \quad 0 \leqslant t \leqslant 1. \tag{4}$$

Eq. (4) was introduced in [9], and we will refer to it as Kummer's equation. The form of (4) and the appearance of λ in it suggest that its solutions will be oscillatory — and most of them are. However, Bessel's equation

$$y''(t) + \left(1 - \frac{\lambda^2 - 1/4}{t^2}\right) y(t) = 0 \quad \text{for all} \quad 0 < t < \infty \tag{5}$$

furnishes a nontrivial example of an equation which admits a nonoscillatory phase function regardless of the value of λ. If we define u, v by the formulas

$$u(t) = \sqrt{\frac{\pi t}{2}} J_\lambda(t) \tag{6}$$

and

$$v(t) = \sqrt{\frac{\pi t}{2}} Y_\lambda(t), \tag{7}$$

where J_λ and Y_λ denote the Bessel functions of the first and second kinds of order λ, and let α be defined by the relations (2), (3), then

$$\alpha'(t) = \frac{2}{\pi t}\frac{1}{J_\lambda^2(t) + Y_\lambda^2(t)}. \tag{8}$$

It can be easily verified that (8) is nonoscillatory. The existence of this nonoscillatory phase function for Bessel's equation is the basis of several methods for the evaluation of Bessel functions of large orders and for the computation of their zeros [6,8,15].

The general situation is not quite so favorable: there need not exist a nonoscillatory function α such that (2) and (3) are exact solutions of (1). However, assuming that q is nonoscillatory and λ is sufficiently large, there exists a nonoscillatory function α such that (2), (3) approximate solutions of (1) with spectral accuracy (i.e., the approximation errors decay exponentially with λ).

To see that this claim is plausible, we apply Newton's method for the solution of nonlinear equations to Kummer's equation (4). In doing so, it will be convenient to move the setting of our analysis from the interval $[0, 1]$ to the real line so that we can use the Fourier transform to quantity the notion of "nonoscillatory." Suppose that the extension of q to the real line is smooth and strictly positive, and such that $\log(q)$ is a smooth function with rapidly decaying Fourier transform. Letting

$$\left(\alpha'(t)\right)^2 = \lambda^2 \exp(r(t)) \tag{9}$$

in (4) yields the logarithm form of Kummer's equation:

$$r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2\left(\exp(r(t)) - q(t)\right) = 0 \quad \text{for all} \quad t \in \mathbb{R}. \tag{10}$$

We use $\{r_n\}$ to denote the sequence of Newton iterates for Eq. (10) obtained from the initial guess

$$r_0(t) = \log(q(t)). \tag{11}$$

The function r_0 corresponds to the first order WKB approximations for (1). That is to say that if we insert the associated phase function

$$\alpha_0(t) = \lambda \int_0^t \exp\left(\frac{1}{2} r_0(u)\right) du = \lambda \int_0^t \sqrt{q(u)} du \tag{12}$$

into (2), (3), then

$$u(t) = q^{-1/4}(t) \cos\left(\lambda \int_0^t \sqrt{q(u)}\, du\right) \tag{13}$$

and