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Angular momentum preserving cell-centered Lagrangian and Eulerian schemes on arbitrary grids

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ABSTRACT

We address the conservation of angular momentum for cell-centered discretization of compressible fluid dynamics on general grids. We concentrate on the Lagrangian step which is also sufficient for Eulerian discretization using Lagrange+Remap. Starting from the conservative equation of the angular momentum, we show that a standard Riemann solver (a nodal one in our case) can easily be extended to update the new variable. This new variable allows to reconstruct all solid displacements in a cell, and is analogous to a partial Discontinuous Galerkin (DG) discretization. We detail the coupling with a second-order Muscl extension. All numerical tests show the important enhancement of accuracy for rotation problems, and the reduction of mesh imprint for implosion problems. The generalization to axi-symmetric case is detailed.

1. Introduction

This work intends to contribute to a long lasting CFD debate which is the enhancement of the accuracy of compressible fluid solvers for vortical flow. In our case we concentrate more on cell-centered Lagrangian compressible schemes on moving grids. But as demonstrated by the numerical results, the proposed approach is also valid for Eulerian calculations on a fixed grid using a Lagrange+Remap procedure.

A seminal and inspirational work is the one of Dukowicz and Meltz [19] where the authors analyze the spurious vorticity errors of the Lagrangian Caveat scheme [1] and develop a procedure to correct these errors. It is in our mind representative of situations where vorticity of the numerical flow is seen as a potential source of problems, that must be controlled. Such kind of procedure has also been developed in [9] for staggered schemes (the curl-Q pseudo-viscosity). A general review of vorticity in Finite Volume schemes is in [36]. See also [39].

In our case we consider that the situation of cell-centered Lagrangian schemes has somewhat changed since the Dukowicz–Meltz contribution. Cell-centered Lagrangian are now becoming a mature ensemble of techniques, owing to the preservation of the GCL (geometric conservation law) and the compatibility with the entropy principle which is rendered possible by the use of nodal-based Riemann solvers instead of the standard edge-based solvers used in [19]. It started in [16,17], and was developed in [31]. The numerical formalism of the general multi-D Glace scheme is developed in [10]. The Eucclhyd formalism on 2D grids was later proposed in [32]. The difference between the nodal Riemann solvers of Glace and Eucclhyd is quite small, similar in a sense to different quadrature formulas in the theory of Finite Element Methods for

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elliptic equations. We quote [11,6,39,2,4] for recent related works for compressible Lagrangian fluid dynamics. It has also been used to develop artificial viscosities for staggered schemes [7,28]. Extension to elastoplastic solvers has been recently performed in a series of papers: it started with [25] where the form of the nodal elastic Riemann solver is defined in the context of very general hyper-elastic models, then later extended in [33,8] for simplified hypoelastic models. Definition of a stabilization procedure named subzonal entropy is proposed in [15]. A proof of weak consistency is given in [14]. Most of these methods have a wide domain of efficiency in terms of stability and accuracy, the main reason being the compatibility with the GCL and with the entropy principle. Moreover cell-centered Lagrangian schemes are naturally adapted to remapping procedure which means that any stability issue of the mesh can be addressed using ALE (Arbitrary Lagrange Euler), still maintaining the conservation properties and the accuracy for shock calculations, see for example [20,13,6]. All these Lagrangian schemes can be run in a purely Eulerian mode by using Remap at every time steps. Based on these advances we consider that the time is less to consider that vorticity is a spuriously that must be controlled or eliminated (another drawback being of course that physical vorticity might be treated like spurious vorticity), but more to enhance the accuracy of flow with strong vortical parts.

Our analysis starts from a well known physical principle which is that angular momentum

$$\mathbf{w} = \mathbf{u} \wedge \mathbf{x} \tag{1}$$

is solution of a conservation law

$$\partial_t(\rho \mathbf{w}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} \wedge \mathbf{x}) + \mathbf{curl}(p\mathbf{x}) = 0. \tag{2}$$

At the analytical level this law is redundant with the inertial momentum equation

$$\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0. \tag{3}$$

But at the numerical level, many basic simulations show without any doubt that angular momentum is far to be preserved by standard cell-centered flow solvers on general grids (situation is less severe on Cartesian fixed grids). This is why we concentrate in this work on the development and analysis of a general numerical method for the preservation of the angular momentum variable (1). We will show that angular momentum discretization can be understood as a special DG discretization of (3), which seems to us a new result with respect to the literature [29,40–42]. We notice also that the vorticity is easily recovered from the knowledge of angular momentum and of the inertial momentum. Let $\mathbf{x}_0$ be a given point in the domain and consider $\mathbf{w}_0(\mathbf{x}) = \mathbf{w}(\mathbf{x}) - \mathbf{u}(\mathbf{x}) \wedge \mathbf{x}_0 = \mathbf{u}(\mathbf{x}) \wedge (\mathbf{x} - \mathbf{x}_0)$. Then $\Delta \mathbf{w}_0(\mathbf{x}) = \Delta u(\mathbf{x}) \wedge (\mathbf{x} - \mathbf{x}_0) - 2\nabla \wedge \mathbf{u}(\mathbf{x})$. Therefore one has the identity $\nabla \wedge \mathbf{u}(\mathbf{x}_0) = -\frac{1}{2}\Delta \mathbf{w}_0(\mathbf{x})$ which shows that the vorticity

$$\omega = \nabla \wedge \mathbf{u}$$

can be computed once one knows the inertial momentum and the angular momentum.

Conservation of angular momentum is also an important question for many different fundamental problems in fluid dynamics. We just give few examples. A first problem is fluid simulations of the atmosphere around the earth. Indeed it is known that angular momentum of the atmosphere may interact with the angular momentum of the planet itself, an early work on this topic is to be found in [35]. Quoting indeed a recent PhD thesis [18], *the Angular Momentum budget represents a beautiful example of how the atmosphere, oceans and solid earth interact.* In this context an accurate computation of angular momentum is necessary to simulate such systems with fluid flow solvers. For this example we are not aware of any use of standard Finite Volume CFD schemes. A completely different physical problem is rotation of MHD flows in Tokamaks for which angular preservation is clearly fundamental issue. It is addressed in the context of MHD solvers, either full MHD or reduced MHD, a general review is to found in [23]. We notice that Finite Volume techniques are rarely used in the Tokamaks community. On the other hand Godunov solvers are widely used for astrophysical flows, and angular momentum is a key feature for an accurate numerical treatment of the rotation of stars and planets: many works are devoted to this issue on Cartesian fixed grids and we quote only on few of them such as [34,37]. In this context, Käppeli and Mishra have recently proposed a Godunov scheme in Eulerian frame to address the issue of angular momentum conservation [24]. It has also a big impact for the chemical reactions into the combustion chamber of engines, in which the intake valve is usually placed to give the mixture a pronounced swirl [3]. The initial stage of turbulent flows is also clearly dominated by the strong vortices inside the flow. In the context of this work, we will show that preservation of the angular momentum enhances a lot the accuracy of implosion calculations near the focusing point, and that it minimizes the mesh imprint for such problems. A simple proof will be given that explains this fact. We stop here the list of such examples, but it is clear that vortical flows and related problems challenge the quality of flow solvers on arbitrary grids in many areas of applied science and computational fluid dynamics.

The plan of the works is as follows. The basis of our method is detailed in Section 2, where we propose to add a local degree of freedom to respect the preservation of angular momentum. The structure of the new scheme is detailed using the Glace formalism. We also explain how the new scheme can be recast as a special DG method. Next in Section 3, we analyze the stability with the entropy principle. Section 4 is devoted to some key implementation details, in particular how to design an angular momentum scheme compatible with the Muscl techniques which are in our case essential to obtain a stable second-order. Implementation of the method in axi-symmetric formulation is addressed in Section 5. We then turn to dedicated numerical examples in Section 6 and conclude.