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Short note

Entropy stable discontinuous interfaces coupling for the three-dimensional compressible Navier–Stokes equations

Matteo Parsani *, Mark H. Carpenter, Eric J. Nielsen

Computational AeroSciences Branch, NASA Langley Research Center (LaRC), Hampton, VA 23681, USA

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1. Introduction

Non-linear entropy stability and a summation-by-parts (SBP) framework are used to derive entropy stable interior interface coupling for the semi-discretized three-dimensional (3D) compressible Navier–Stokes equations. A complete semi-discrete entropy estimate for the interior domain is achieved combining a discontinuous entropy conservative operator of any order [1,2] with an entropy stable coupling condition for the inviscid terms, and a local discontinuous Galerkin (LDG) approach with an interior penalty (IP) procedure for the viscous terms. The viscous penalty contributions scale with the inverse of the Reynolds number (Re) so that for $Re \to \infty$ their contributions vanish and only the entropy stable inviscid interface penalty term is recovered. This paper extends the interface couplings presented [1,2] and provides a simple and automatic way to compute the magnitude of the viscous IP term. The approach presented herein is compatible with any diagonal norm summation-by-parts (SBP) spatial operator, including finite element, finite volume, finite difference schemes and the class of high-order accurate methods which include the large family of discontinuous Galerkin discretizations and flux reconstruction schemes.

This note relies on the formalism introduced in [1,3] and complements the new class of interior entropy stable SBP operators of any order for the 3D compressible Navier–Stokes equations on unstructured grids that was proposed in [1,2]. To keep the notation as simple as possible, a uniform Cartesian grid is considered in the derivation. However, the extension to generalized curvilinear coordinates and unstructured grids follows immediately if the transformation from computational to physical space preserves the semi-discrete geometric conservation [4].

The proposed interface coupling technique has been successfully combined with a high order entropy stable discretization for the simulation of two-dimensional (2D) and 3D viscous subsonic and supersonic flows presented in [1,5,6].

* Corresponding author.
E-mail addresses: matteo.parsani@nasa.gov (M. Parsani), mark.h.carpenter@nasa.gov (M.H. Carpenter), eric.j.nielsen@nasa.gov (E.J. Nielsen).

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2. Compressible Navier–Stokes equations and entropy function

Consider a fluid in a domain Ω with boundary surface denoted by $\partial\Omega$, without radiation and external volume forces. In this context, the compressible Navier–Stokes equations, equipped with suitable boundary and initial conditions, may be expressed in the form

$$
\begin{aligned}
&\frac{\partial q}{\partial t}+\frac{\partial f_i^{(I)}}{\partial x_i}=\frac{\partial f_i^{(V)}}{\partial x_i}, \quad x\in\Omega, \quad t\in[0,\infty),\\
&q|_{\partial\Omega}=g^{(B)}(x,t), \quad x\in\partial\Omega, \quad t\in[0,\infty),\\
&q(x,0)=g^{(0)}(x), \quad x\in\Omega,
\end{aligned}
\tag{1}
$$

where the Cartesian coordinates, $x=(x_1,x_2,x_3)^\top$, and time, t, are the independent variables. Note that in (1) Einstein notation is used. The vectors $q(x,t)$, $f_i^{(I)}=f_i^{(I)}(q)$, and $f_i^{(V)}=f_i^{(V)}(q,\nabla q)$ are the conserved variables, and the inviscid and viscous fluxes in the i direction, respectively.[1] Both boundary conditions, $g^{(B)}$, and initial data, $g^{(0)}$, are assumed to be bounded, $L^2\cap L^\infty$. Furthermore, $g^{(B)}$ is also assumed to contain linearly well-posed data. The conservative variable vector is $q=(\rho,\rho u_1,\rho u_2,\rho u_3,\rho E)^\top$, where ρ denotes the density, $u=(u_1,u_2,u_3)^\top$ is the velocity vector, and E is the specific total energy.

Harten [7] and Tadmor [8] showed that systems of conservation laws are symmetrizable if and only if they are equipped with a convex mathematical entropy function, $S(q)$. Given a set of conservation variables $q(x,t)$, the entropy variables which symmetrize the system are defined as the derivatives of the mathematical entropy function with respect to $q(x,t)$, $\partial S/\partial q$. Hughes and co-authors [9] extended these ideas to the compressible Navier–Stokes equations (1). Therein, it is shown that the mathematical entropy must be an affine function of the physical (or thermodynamic) entropy function and that semi-discrete solutions obtained from a weighted residual formulation based on entropy variables will respect the second law of thermodynamics. Hence, it is again found that the entropy function and the entropy variables are critical ingredients in the design of numerical schemes exhibiting non-linear stability.

In the specific case of the compressible Navier–Stokes equations (1), the entropy function is defined as $S=S(q)=-\rho s$, where s is the thermodynamic entropy. In the entropy analysis that will follow, the definition of the thermodynamic entropy for a perfect gas is the explicit form,

$$
s=\frac{R}{\gamma-1}\log\left(\frac{T}{T_\infty}\right)-R\log\left(\frac{\rho}{\rho_\infty}\right),
\tag{2}
$$

where R, γ, T, T_∞, and ρ_∞ are the gas constant, the ratio of the heat capacity at constant pressure c_p to heat capacity at constant volume c_v, the temperature, and the reference temperature and density, respectively.

The scalar function $S=-\rho s$ satisfies the following conditions (see, for instance, [1,3] and the references therein):

- The function $S(q)$ when differentiated with respect to the conservative variables (i.e., $\partial S/\partial q$) simultaneously contracts all the inviscid spatial fluxes as follows

$$
\frac{\partial S}{\partial q}\frac{\partial f_i^{(I)}}{\partial x_i}=\frac{\partial S}{\partial q}\frac{\partial f_i^{(I)}}{\partial q}\frac{\partial q}{\partial x_i}=\frac{\partial F_i}{\partial q}\frac{\partial q}{\partial x_i}=\frac{\partial F_i}{\partial x_i}, \quad i=1,2,3.
\tag{3}
$$

  The components of the contracting vector, $\partial S/\partial q$, are the entropy variables defined as $w^\top=\partial S/\partial q=\left(\frac{h}{T}-s-\frac{u_iu_i}{2T},\frac{u_1}{T},\frac{u_2}{T},\frac{u_3}{T},-\frac{1}{T}\right)$, where h denotes the specific entalphy which is defined as $h=c_pT$ for a perfect gas. $F_i(q)=-\rho u_i s$, $i=1,2,3$, are the entropy fluxes in the three Cartesian directions (see, for instance, [10]).
- The new entropy variables, w, symmetrize the system of equations (1):

$$
\frac{\partial q}{\partial t}+\frac{\partial f_i^{(I)}}{\partial x_i}-\frac{\partial f_i^{(V)}}{\partial x_i}=\frac{\partial q}{\partial w}\frac{\partial w}{\partial t}+\frac{\partial f_i^{(I)}}{\partial w}\frac{\partial w}{\partial x_i}-\frac{\partial}{\partial x_i}\left(\widehat{c}_{ij}\frac{\partial w}{\partial x_j}\right)=0, \quad i=1,2,3
\tag{4}
$$

  with: $\partial q/\partial w=(\partial q/\partial w)^\top$, $\partial f_i^{(I)}/\partial w=\left(\partial f_i^{(I)}/\partial w\right)^\top$ and $\widehat{c}_{ij}=\widehat{c}_{ij}^\top$. The matrices $\widehat{c}$ are positive semi-definite [11].
- The function $S(q)$ is convex, meaning that the Hessian, $\partial^2S/\partial q^2=\partial w/\partial q$, is symmetric positive definite,

$$
\zeta^T\frac{\partial^2 S}{\partial q^2}\zeta>0, \quad \forall\zeta\neq 0,
\tag{5}
$$

  and yields a one-to-one mapping from conservation variables, q, to entropy variables, $w^\top=\partial S/\partial q$. A sufficient (and also physical) condition to ensure the convexity of $S(q)$ is that $\rho, T>0$ [9,11].

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[1] The symbol ∇q denotes the gradient of the conservative variables.