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Numerical solution of an inverse obstacle scattering problem with near-field data ☆

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ABSTRACT

Consider the scattering of an arbitrary time-harmonic incident wave by a sound soft obstacle. In this paper, a novel method is presented for solving the inverse obstacle scattering problem of the two-dimensional Helmholtz equation, which is to reconstruct the obstacle surface by using the near-field data. The obstacle is assumed to be a small and smooth perturbation of a disc. The method uses the transformed field expansion to reduce the boundary value problem into a successive sequence of one-dimensional problems which are solved in closed forms. By dropping the higher order terms in the power series expansion and truncating the infinite linear system for the first order term, the inverse problem is linearized and an approximate but explicit formula is obtained between the Fourier coefficients of the solution and data. A nonlinear correction algorithm is introduced to improve the accuracy of the reconstructions for large deformations. Numerical examples show that the method is simple, efficient, and stable to reconstruct the obstacle with subwavelength resolution.

1. Introduction

Obstacle scattering problems are concerned with the effect that a bounded scatterer has on an incident field. These problems are fundamental in a wide range of applications [9,12,13] such as radar and sonar, geophysical exploration, medical imaging, nondestructive testing, and near-field optics. Given the incident wave, the direct obstacle scattering problem is to determine the scattered wave for the known obstacle. This paper is concerned with the inverse obstacle scattering problem, which is to reconstruct its surface from the field measured on a circle surrounding the obstacle.

The inverse problem is challenging due to its high nonlinearity and severe ill-posedness. A number of numerical methods have been proposed to overcome the issues. They can be broadly classified into two categories: nonlinear optimization based iterative methods [15,18,20,21] and imaging based direct methods [8,11,17,19,24,25]. The iterative methods require good initial guesses and are computationally expensive as a sequence of direct and adjoint problems need to be solved at each step of iterations. The direct methods require no a priori information on the obstacles and are computationally efficient, but the reconstructions may not be as accurate as those iterative methods.

It is known that conventional reconstruction methods cannot achieve super-resolution because of using only far-field data, i.e., data measured at a distance which is a few wavelengths or longer away from the obstacle. According to the Rayleigh criterion, the resolution of far-field imaging is limited roughly by one half the wavelength of the incident field

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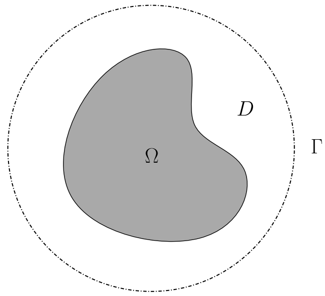

Fig. 1. Geometry of the problem. The obstacle is represented by the domain Ω and the measurement is taken at the circle Γ.

[13], which is known as the diffraction limit. By collecting data in the near-field zone, one may break the diffraction limit and obtain images with subwavelength resolution. This is referred to as near-field imaging [13] and has led to emerging applications in modern science and technology such as nanotechnology, biology, information storage, and surface chemistry.

Recently a novel approach has been developed for solving a variety of inverse scattering problems in near-field imaging, which include infinite rough surfaces [2,3], diffraction gratings [1,10], obstacles [22], and interior cavities [23]. The method begins with the transformed field expansion, which reduces the high dimensional boundary value problems to a successive sequence of one-dimensional two-point boundary value problems in the frequency domain, where analytical solutions are obtained in closed forms. By dropping higher order terms in the power series expansion, the inverse problems are linearized and explicit reconstruction formulas are obtained. The method requires only one incident field at a fixed angle and frequency. Numerical experiments have shown that the method is simple, efficient, and stable to reconstruct the surfaces with subwavelength resolution. We refer to [16] for a related three-dimensional inverse obstacle scattering problem by using the factorization method and to [5–7] for the inverse problems in near-field imaging of local displacement on an infinite ground plane.

In [22], the incident field is assumed to be an incoming cylindrical wave. Mathematically, it is a good choice in view of the problem geometry as it facilitates the analysis and leads to an explicit reconstruction formula. However, this special incident wave may not be easily realized in practice. In this work, we extend the method to arbitrary time harmonic incident waves which include the commonly used plane wave and the point source wave. This nontrivial extension makes the method more universal in a wide range of practical situations. Although we focus on the two-dimensional Helmholtz equation in this work, the method can be also extended to deal with the three-dimensional Helmholtz and Maxwell's equations.

The outline of the paper is as follows. In Section 2, the inverse obstacle scattering problem is introduced and a simple uniqueness result is given. Section 3 is devoted to the transformed field expansion for the boundary value problem whose solution is obtained in a closed form. The reconstruction formula is derived and a nonlinear correction algorithm is described in Section 4. Numerical results are presented in Section 5. The paper is concluded with general remarks and directions for future research in Section 6.

2. The inverse scattering problem

As seen in Fig. 1, let the obstacle be a small and smooth perturbation of a disk in $\mathbb{R}^2$. In the polar coordinate, it can be described by a domain

$$\Omega = \{(r,\theta) : 0 < r < a + f(\theta), \theta \in [0, 2\pi)\},$$

where $a > 0$ is a constant representing the radius of the unperturbed disk and $f(\theta)$ is the obstacle surface function. We assume that $f \in C^2(\mathbb{R})$ is a 2π-periodic function whose infinity norm is small comparing to the wavelength λ of the incident field. Hence it can be written as

$$f(\theta) = \varepsilon g(\theta),$$

where $\varepsilon > 0$ is a small perturbation parameter and g is the obstacle profile function such that $\|g\|_\infty = \mathcal{O}(\lambda)$.

In the exterior domain $\mathbb{R}^2 \setminus \bar{\Omega}$, the space is assumed to be filled with a homogeneous medium characterized by a constant wavenumber $\kappa = 2\pi/\lambda$. An incoming wave $u^{\rm inc}(r,\theta)$ is incident on the obstacle and generates the scattered wave $u^{\rm sca}$. Clearly the total field u consists of the incident field and the scattered field, i.e., $u = u^{\rm inc} + u^{\rm sca}$. For a sound soft obstacle, the total field vanishes on the obstacle surface, i.e.,

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Given the incident field $u^{\rm inc}$ and the obstacle surface function f, the direct problem is to determine the total field u. This paper is focused on the inverse problem: given the incident field $u^{\rm inc}$, reconstruct the surface function f from the total field u measured at a circle

$$\Gamma = \{(b,\theta) : r = b > a + \|f\|_\infty, \theta \in [0, 2\pi)\}.$$