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A primal–dual mimetic finite element scheme for the rotating shallow water equations on polygonal spherical meshes

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ABSTRACT

A new numerical method is presented for solving the shallow water equations on a rotating sphere using quasi-uniform polygonal meshes. The method uses special families of finite element function spaces to mimic key mathematical properties of the continuous equations and thereby capture several desirable physical properties related to balance and conservation. The method relies on two novel features. The first is the use of *compound finite elements* to provide suitable finite element spaces on general polygonal meshes. The second is the use of *dual finite element spaces* on the dual of the original mesh, along with suitably defined discrete Hodge star operators to map between the primal and dual meshes, enabling the use of a finite volume scheme on the dual mesh to compute potential vorticity fluxes. The resulting method has the same mimetic properties as a finite volume method presented previously, but is more accurate on a number of standard test cases.

1. Introduction

In order to exploit the new generation of massively parallel supercomputers that are becoming available, weather and climate models will require good parallel scalability. This requirement has driven the development of numerical methods that do not depend on the orthogonal coordinate system and quadrilateral structure of the longitude–latitude grid, whose polar resolution clustering is predicted to lead to a scalability bottleneck. A significant challenge is to obtain good scalability without sacrificing accuracy; in particular conservation, balance, and wave propagation are important for accurate modeling of the atmosphere [24].

Building on earlier work [23,26], Thuburn et al. [27] presented a finite volume scheme for the shallow water equations on polygonal meshes. They start from the continuous shallow water equations in the so-called vector invariant form:

$$\phi_t + \nabla \cdot \mathbf{f} = 0, \tag{1}$$

$$\mathbf{u}_t + \mathbf{q}^\perp + \nabla(\phi_\mathrm{T} + k) = 0, \tag{2}$$

where ϕ, the geopotential, is equal to the fluid depth times the gravitational acceleration, $\phi_\mathrm{T} = \phi + \phi_\mathrm{orog}$ is the total geopotential at the fluid's upper surface including the contribution from orography, $\mathbf{u}$ is the velocity, $\mathbf{f} = \mathbf{u}\phi$ is the mass flux, and $k = |\mathbf{u}|^2/2$. The $\perp$ symbol is defined by $\mathbf{u}^\perp = \mathbf{k} \times \mathbf{u}$ where $\mathbf{k}$ is the unit vertical vector. Finally, $\mathbf{q} = \mathbf{f}\pi$ is the flux of potential vorticity (PV), where $\pi = \zeta/\phi$ is the PV and $\zeta = f + \xi$ is the absolute vorticity, with f the Coriolis parameter

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and $\xi = \mathbf{k} \cdot \nabla \times \mathbf{u}$ the relative vorticity. (The notation used in the paper is summarized in Appendix B.) By the use of a C-grid placement of prognostic variables, and by ensuring that the numerical method mimics key mathematical properties of the continuous governing equations (hence the term 'mimetic'), the scheme was designed to have good conservation and balance properties. These good properties were verified in numerical tests on hexagonal and cubed sphere spherical meshes. However, their scheme has a number of drawbacks. Most seriously, the Coriolis operator, whose discrete form is essential to obtaining good geostrophic balance, is numerically inconsistent and fails to converge in the L_∞ norm [31,27]. Also, although the gradient and divergence operators are consistent, their combination to form the discrete Laplacian operator also fails to converge in the L_∞ norm in some cases. These inaccuracies are clearly visible in idealized convergence tests, and give rise to marked 'grid imprinting' for initially symmetrical flows. Although they are less conspicuous in more complex flows, they are clearly undesirable.

Cotter and Shipton [9] (see also McRae and Cotter [17], Cotter and Thuburn [10]) showed that the same mimetic properties can be obtained using a certain class of mixed finite element method. The mimetic properties follow from the choice of an appropriate hierarchy of function spaces for the prognostic and diagnostic variables (e.g. Section 3 below), which also provides a finite element analogue of the C-grid placement of variables, or a higher-order generalization. (The use of such a hierarchy goes by various names in the literature, including 'mimetic finite elements', 'compatible finite elements', and 'finite element exterior calculus'; see Cotter and Thuburn [10] for a discussion of the shallow water equation case in the language of exterior calculus.) Importantly, the resulting schemes are numerically consistent.

While the mimetic finite element approach appears very attractive, it is not yet clear which particular choice of mesh and function spaces is most suitable. Standard finite element methods use triangular or quadrilateral elements. For the lowest-order mimetic finite element scheme on triangles, the dispersion relation for the linearized shallow water equations suffers from extra branches of inertio-gravity waves, which are badly behaved numerical artefacts [15], analogous to the problem that occurs on the triangular C-grid [11]. Higher-order finite element methods also typically exhibit anomalous features in their wave dispersion relations, such as extra branches, frequency gaps, or zero group velocity modes. Some progress has been made in reducing these problems, at least on quadrilateral meshes, through the inclusion of dissipation or modification of the mass matrix (e.g. Ullrich [29], Melvin et al. [18]), though the remedies are somewhat heuristic except in the most idealized cases. Finally, coupling to subgrid models of physical processes such as cumulus convection or cloud microphysics may be less straightforward with higher-order elements (P. Lauritzen, pers. comm.).[1] These factors suggest that it may still be worthwhile investigating lowest-order schemes on quadrilateral and hexagonal meshes.

The above arguments raise two related questions. Can the mimetic finite element method inspire a development to fix the inconsistency of the mimetic finite volume method? Alternatively, can the mimetic finite element method at lowest order be adapted to work on polygonal meshes such as hexagons? Below we answer the second question by showing that the mimetic finite element method can indeed be adapted. In fact, from a certain viewpoint the mimetic finite volume and mimetic finite element schemes have very similar mathematical structure. The notation below is chosen to emphasize this similarity.[2] Moreover, the similarity is sufficiently strong that much of the code of the mimetic finite volume model of Thuburn et al. [27] could be re-used in the model presented below. This, in turn, facilitates the cleanest possible comparison of the two approaches.

The adaptation of the mimetic finite element method employs two novel features. The first is the definition of a suitable hierarchy of finite element function spaces on polygonal meshes. Following Christiansen [7], this is achieved by defining compound elements built out of triangular subelements, and is described in Section 3. The second ingredient is the introduction of a dual family of function spaces that are defined on the dual of the original mesh. This permits the definition of a spatially averaged mass field that lives in the same function space as the vorticity and potential vorticity fields; this, in turn, enables the use of an accurate finite volume scheme on the dual mesh for advection of potential vorticity, and keeps the formulation of the finite element model as close as possible to that of the finite volume model.

2. Meshes and dual meshes

The scheme described here is suitable for arbitrary two-dimensional polygonal meshes on flat domains or, as used here, curved surfaces approximated by planar facets. Two particular meshes are used to obtain the results in Section 5, namely the same variants of the hexagonal–icosahedral mesh and the cubed sphere mesh used by Thuburn et al. [27], in order to facilitate comparison with their results. The hexagonal–icosahedral mesh uses the Heikes and Randall [13] optimization, which greatly improves the accuracy of the Laplacian for a finite volume scheme but has little effect for the present finite element scheme. The cubed sphere mesh is based on an equiangular cubed sphere, modified so that vertices are located at the barycenters of the surrounding cell centers. Again, this modification significantly improves the accuracy of a finite

[1] The issues include, but are not limited to, (i) uneven spacing of the collocation points, especially in the vertical; (ii) whether the degrees of freedom should be treated as pointwise values, and, for discontinuous Galerkin schemes in particular, how to handle the case in which the data are discontinuous at an element boundary node; (iii) how to ensure positivity of fields like specific humidity; (iv) consistency of operators like divergence between the subgrid model and the dynamical core.

[2] Readers wishing to compare the two formulations should note that a different sign convention is used for the expansion coefficients of $\mathbf{k}\times$ any vector, such as $U^\perp$ in (48).