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Multidimensional Riemann problem with self-similar internal structure. Part II – Application to hyperbolic conservation laws on unstructured meshes

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ABSTRACT

Multidimensional Riemann solvers that have internal sub-structure in the strongly-interacting state have been formulated recently (D.S. Balsara (2012, 2014) [5,16]). Any multidimensional Riemann solver operates at the grid vertices and takes as its input all the states from its surrounding elements. It yields as its output an approximation of the strongly interacting state, as well as the numerical fluxes. The multidimensional Riemann problem produces a self-similar strongly-interacting state which is the result of several one-dimensional Riemann problems interacting with each other. To compute this strongly interacting state and its higher order moments we propose the use of a Galerkin-type formulation to compute the strongly interacting state and its higher order moments in terms of similarity variables. The use of substructure in the Riemann problem reduces numerical dissipation and, therefore, allows a better preservation of flow structures, like contact and shear waves. In this second part of a series of papers we describe how this technique is extended to unstructured triangular meshes. All necessary details for a practical computer code implementation are discussed. In particular, we explicitly present all the issues related to computational geometry. Because these Riemann solvers are ***Mu***ltidimensional and have ***S***elf-similar strongly-***I***nteracting states that are obtained by ***C***onsistency with the conservation law, we call them **MuSIC** Riemann solvers. (A video introduction to multidimensional Riemann solvers is available on http://www.nd.edu/~dbalsara/Numerical-PDE-Course.)

The MuSIC framework is sufficiently general to handle general nonlinear systems of hyperbolic conservation laws in multiple space dimensions. It can also accommodate all self-similar one-dimensional Riemann solvers and subsequently produces a multidimensional version of the same. In this paper we focus on unstructured triangular meshes. As examples of different systems of conservation laws we consider the Euler equations of compressible gas dynamics as well as the equations of ideal magnetohydrodynamics (MHD). Several stringent test problems are solved for both PDE systems.

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1. Introduction

Riemann problems are crucially important to the solution of hyperbolic conservation laws. The Riemann solvers mimic the self-similarity of the original Riemann problem, though that property has not seen much use in the design of one-dimensional Riemann solvers. Introducing multidimensionality into the solution of the Riemann problem also has great utilitarian promise. A prior paper (Balsara [16]) has already shown that self-similarity is very useful in designing multidimensional Riemann solvers for structured meshes. This paper shows that the concept of self-similarity is also very useful in designing multidimensional Riemann solvers for hyperbolic conservation laws on unstructured meshes. By now, several Riemann solvers have been presented in the literature that are based on incorporating true multidimensionality along with a self-similar, strongly-interacting state that is obtained via consistency with the hyperbolic system (Balsara [4,5,16], Balsara, Dumbser and Abgrall [15]). Because they share the philosophy of being ***Mu***ltidimensional with ***S***elf-similar strongly-***I***nteracting states that are obtained via ***C***onsistency, we call them **MuSIC** (Multidimensional, Self-similar, strongly-Interacting, Consistent) Riemann solvers. That name was first suggested in Balsara [16] and we continue the nomenclature here.

Godunov [37,38] and van Leer [61] showed the importance of the one-dimensional Riemann solver for any Godunov-type scheme. Exact (van Leer [61]) and the approximate (Colella [25], Colella and Woodward [27]) Riemann solvers have been formulated. See also the work of Chorin [23]. Osher and Solomon [47] and Dumbser and Toro [31] presented approximate Riemann solvers whose numerical dissipation term is based on a path integral in phase space. The linearized Riemann solver by Roe [48] and the HLL/HLLE/HLLEM Riemann solvers (Harten, Lax and van Leer [40], Einfeldt [32], Einfeldt et al. [33]) and the local Lax–Friedrichs (LLF) Riemann solver (Rusanov [52]) are also frequently used. The HLLEM scheme was proposed in [33], where the constant intermediate state of the original HLLE method was substituted by a piecewise linear distribution that allowed to resolve the contact wave exactly. Toro, Spruce and Speares [58–60], Chakraborty and Toro [22] and Batten et al. [18] showed how the contact discontinuity can be reintroduced in the HLL-type Riemann solvers, yielding a family of HLLC Riemann solvers. See also, Billett and Toro [19]. In Balsara [16] we showed that all of the above-mentioned self-similar hydrodynamical Riemann solvers can serve as building blocks for the multidimensional Riemann solvers if the problem is viewed in similarity variables. This was done for structured meshes and the *first goal* of this paper is to show that an analogous result can be obtained on unstructured meshes.

Magnetohydrodynamics (MHD) is an interesting example of a conservation law with anisotropic wave motion. One-dimensional linearized Riemann solvers for numerical MHD have been designed (Roe and Balsara [50], Cargo and Gallice [21], Balsara [6]). HLLC Riemann solvers, capable of capturing mesh-aligned contact discontinuities, have been presented by Gurski [39] and Li [43]. Miyoshi and Kusano [45] extended the work of Gurski and designed the so-called HLLD solver, which is also able to capture the linear degenerate Alfvén waves of the MHD system. The MHD equations constitute a very interesting nonlinear hyperbolic system of conservation laws, since they contain multiple wave foliations even in the one-dimensional Riemann problem. The *second goal* of this paper is therefore to accommodate more complicated one-dimensional Riemann solvers with multiple wave foliations within the framework of multidimensional Riemann solvers. We show that this can be accomplished by using similarity variables.

Several authors have tried to build some level of multidimensionality out of the one-dimensional Riemann solvers (Colella [26], Saltzman [53], LeVeque [42]). Early attempts to build genuinely multidimensional Riemann solvers (Roe [49], Rumsey, van Leer and Roe [51]) ended in failure.

Abgrall [1,2] formulated a genuinely multidimensional Riemann solver for CFD that worked. Further advances were also reported (Fey [34,35], Gilquin, Laurens and Rosier [36], Brio, Zakharian and Webb [20], Lukacsova-Medvidova et al. [44]). The aforementioned multidimensional Riemann solvers have very interesting properties, but most of them are rather difficult to implement and are limited to the Euler equations of gas dynamics, which may have prevented their widespread use in the CFD community so far. Wendroff [63] formulated a two-dimensional HLL Riemann solver, but his method was also not easy to implement. Working on logically rectangular meshes, Balsara [4] presented a two-dimensional HLL Riemann solver with simple closed form expressions for the fluxes that were easy to implement. Balsara [5] subsequently presented a two-dimensional HLLC Riemann solver which incorporated the physics of the contact discontinuity and permitted the contact discontinuity to propagate in any direction relative to the mesh. Balsara, Dumbser and Abgrall [15] extended this formulation to accommodate unstructured meshes. Multidimensional Riemann solvers have subsequently been formulated in self-similarity variables and the advantages of doing so have been documented in Balsara [16] for Cartesian meshes. While the equations for extending this work to unstructured meshes were derived in Section 4 of that same paper, the full extension had not been done in that work. This is because one also needs to build up a fair bit of computational geometry in order to support a multidimensional Riemann solver on unstructured meshes. The *third goal* of this paper is to document the development of such computational geometry for self-similar multidimensional Riemann solvers that operate on unstructured meshes.

Schulz-Rinne, Collins and Glaz [54] studied the multidimensional Riemann problem computationally and showed that the one-dimensional Riemann problems interact amongst each other to form the strongly-interacting state. This strongly interacting state also evolves self-similarly in multi-dimensions. In previous works, Balsara [4,5] and Balsara, Dumbser and Abgrall [15], we identified this strongly-interacting state with the multidimensional HLL resolved state. Imparting a sub-structure to such a state can be desirable, and the strongly-interacting states from Schulz-Rinne, Collins and Glaz [54] do indeed show interesting sub-structures. While Balsara [5] and Balsara, Dumbser and Abgrall [15] showed how a contact discontinuity can be introduced in the strongly-interacting state, in this paper we would like to impart any general form

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