Journal of Computational Physics 285 (2015) 84–87

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Short note

High order approximation of a tokamak edge plasma transport minimal model with Bohm boundary conditions

Sébastian Minjeaud, Richard Pasquetti *

Lab. J.A. Dieudonné, UMR CNRS 7351, & INRIA project CASTOR, Université de Nice-Sophia Antipolis, F-06108 Nice, France

ARTICLE INFO

Article history:
Received 7 October 2014
Accepted 26 December 2014
Available online 16 January 2015

Keywords:
Magnetic confined fusion
Edge plasma
Bohm boundary conditions
Hyperbolic system
Tokamaks

In the recent paper [2], a penalty method is proposed to address the so-called Bohm boundary conditions which are generally imposed to model the limiters, *i.e.* the walls that intercept in a tokamak the magnetic field lines. This approach is validated by considering a simple one-dimensional hyperbolic system, that constitutes a minimal transport model for ionic density and momentum. In [1] this penalty model is discussed and an improved formulation of the penalty method is proposed. More recently, in [6], a similar model is considered but completed with evolution equations for the ionic and electronic temperatures. Again, a penalty method is proposed to enforce the boundary conditions at the walls.

In the present note, we consider the system studied in [1,2] and show that as soon as the solution is smooth, a spectrally accurate approximation can be obtained. To this end, a stabilized spectral element method (SEM) is preferred to the Godunov schemes used in [1,2,6]. Moreover, instead of the penalty approach, which in the multidimensional context can only offer a first order algebraic convergence, we use a direct imposition of the Bohm boundary condition in an explicit time marching. The system studied in [1,2] writes:

$$\partial_t N + \partial_x \Gamma = S_N \tag{1}$$

$$\partial_t \Gamma + \partial_x \left(\frac{\Gamma^2}{N} + N \right) = S_\Gamma \tag{2}$$

with $t \in \mathbb{R}^+$ and $x \in \Omega = (x_{\min}, x_{\max})$ for the time and space variables and where N and Γ stand for the dimensionless ion density and ion momentum, respectively. In this model, the ion and electron temperatures are assumed to be constant, so that when using the perfect gas law, in dimensionless form the pressure is equal to the density. We then recognize the usual conservation equations of mass and momentum, where some source terms S_N and S_Γ are considered. Moreover,

* Corresponding author.
E-mail address: richard.pasquetti@unice.fr (R. Pasquetti).

http://dx.doi.org/10.1016/j.jcp.2014.12.049

the dimensionless velocity equals the Mach number, *i.e.* $M = \Gamma/N$. It is easy to check that the system (1)–(2) is strictly hyperbolic. In convective form it writes:

$$\partial_t \begin{pmatrix} N \\ \Gamma \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -M^2+1 & 2M \end{pmatrix} \partial_x \begin{pmatrix} N \\ \Gamma \end{pmatrix} = \begin{pmatrix} S_N \\ S_\Gamma \end{pmatrix} \tag{3}$$

and so the wave speeds, defined as the eigenvalues of the matrix in (3), equal $\lambda_\pm = M \pm 1$. As is well known, the number of boundary conditions to be considered depends on the number of ingoing waves. Thus, at $x_{\max}$ two boundary conditions are needed if $M < -1$, one boundary is needed if $-1 < M < 1$ and no boundary conditions are needed if $M > 1$.

In order to model the limiter by a particles sink, the Bohm boundary condition states that $M(x_{\max}) \geq 1$ [3]. Note that $M(x_{\max}) = 1$ corresponds to the case where the boundary $x = x_{\max}$ is characteristic, so that it is not clear what should be done. Thus, in [1] it is suggested to slightly modify the boundary condition by imposing $M(x_{\max}) = 1 - \eta$ and $M(x_{\min}) = 1 + \eta$, with a fixed (small) $\eta > 0$ in order to consider a well-posed hyperbolic problem. From our point of view, here this problem is mainly theoretical and probably not relevant as soon as numerical time and space discretizations are involved. Hereafter, we enforce the following Bohm conditions:

$$M(x_{\min}) \leq -1 \quad \text{and} \quad M(x_{\max}) \geq 1. \tag{4}$$

To check this approach in the frame of high order methods, we use in space the SEM, *i.e.* a high order nodal finite element method such that the approximation space, say E_h, contains all C^0 functions whose restriction in each element is a polynomial of degree p, see *e.g.* [4]. In each element the basis functions are Lagrange polynomials based on the Gauss–Lobatto–Legendre (GLL) points and these GLL interpolation points are also used as quadrature points (quadratures are then exact for polynomials of degree $2p - 1$). Moreover, since the problem is hyperbolic, shocks may develop and a stabilization technique is required. We use the Spectral Vanishing Viscosity (SVV) technique introduced in [5,7] (see also [8] in the multi-dimensional context). With (N_h, Γ_h) for the numerical approximations of (N, Γ), we thus consider the following variational formulation:

$$\int_{x_{\min}}^{x_{\max}} \partial_t N_h v_h dx + \int_{x_{\min}}^{x_{\max}} \partial_x \Gamma_h v_h dx + V_N = \int_{x_{\min}}^{x_{\max}} S_{Nh} v_h dx, \quad \forall v_h \in E_h\,, \tag{5}$$

$$\int_{x_{\min}}^{x_{\max}} \partial_t \Gamma_h v_h dx + \int_{x_{\min}}^{x_{\max}} \partial_x \left(\frac{\Gamma_h^2}{N_h} + N_h \right) v_h dx + V_\Gamma = \int_{x_{\min}}^{x_{\max}} S_{\Gamma h} v_h dx, \quad \forall v_h \in E_h \tag{6}$$

where the SVV stabilization terms write:

$$V_N = \epsilon_{hp} \int_{x_{\min}}^{x_{\max}} Q_p(\partial_x N_h) \partial_x v_h dx, \qquad V_\Gamma = \epsilon_{hp} \int_{x_{\min}}^{x_{\max}} Q_p(\partial_x \Gamma_h) \partial_x v_h dx. \tag{7}$$

In these expressions, ϵ_{hp} controls the amplitude of the SVV term and Q_p is the so-called SVV operator whose kernel contains all low frequency Legendre components, say for $k \leq m_p < p$ of the spectral approximation, see *e.g.* [5] for details. In the computations, we have systematically used $\epsilon_{hp} = h/p$, where h is the grid-size, and $m_p = [\sqrt{p}]$ ([.], for nearest integer).

For the integration in time, we use a standard fourth order Runge Kutta scheme (RK4 scheme). Then, to enforce the Bohm conditions we impose at the end of each RK4 step:

$$N_h(x_{\min}) := -\Gamma_h(x_{\min}) \quad \text{if } \frac{\Gamma_h}{N_h}(x_{\min}) > -1; \qquad N_h(x_{\max}) := \Gamma_h(x_{\max}) \quad \text{if } \frac{\Gamma_h}{N_h}(x_{\max}) < 1. \tag{8}$$

On the basis of this SEM approximation, we have considered three different test cases.

Test case 1: This test-case is the one introduced in [2]. The length of the domain is unitary, $\Omega = (0, 1)$. At the initial time $t = 0$ the fluid is at rest, $M_0 = \Gamma_0 = 0$, and the density is constant, $N_0 = 1$. For the source terms one has $S_N = 2$ and $S_\Gamma = 0$. In [2] the asymptotic solution, *i.e.* obtained at $t \gg 1$, is compared to the analytical one, say $(N_\infty, \Gamma_\infty)$, such that:

$$N_\infty = S_N\big(0.5 + \sqrt{x(1-x)}\big), \qquad \Gamma_\infty = S_N(x - 0.5). \tag{9}$$

The computation has been carried out with $K = 20$ elements and with the polynomial degree $p = 6$. In Fig. 1, we show the solution $(N_h, \Gamma_h/N_h)$ at different times. At the final time of the computation, the numerical solution clearly agrees with the analytical one. Despite the use of the SVV technique, one can discern that the solution is not perfectly smooth. This is mainly associated to the fact that for this test case the initial profiles N_0 and Γ_0 do not verify the Bohm conditions, since $M(0) = M(1) = 0$ at $t = 0$.

Test case 2: This test-case is similar to test-case 1, *i.e.* $S_N = 2$ and $S_\Gamma = 0$, but to check our algorithm with a supersonic–subsonic transition at the boundary, the initial condition for the momentum is now $\Gamma_0 = 2(x - 0.5) - 2\sin(\pi x)$. Then, at the initial time, the flow is sonic at the boundaries and partially subsonic and supersonic inside the domain. This situation