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A fast-marching like algorithm for geometrical shock dynamics

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ABSTRACT

We develop a new algorithm for the computation of the Geometrical Shock Dynamics (GSD) model. The method relies on the fast-marching paradigm and enables the discrete evaluation of the first arrival time of a shock wave and its local velocity on a Cartesian grid. The proposed algorithm is based on a first order upwind finite difference scheme and reduces to a local nonlinear system of two equations solved by an iterative procedure. Reference solutions are built for a smooth radial configuration and for the 2D Riemann problem. The link between the GSD model and p-systems is given. Numerical experiments demonstrate the efficiency of the scheme and its ability to handle singularities.

1. Introduction

In 1957, when G.B. Whitham published the Geometrical Shock Dynamics (GSD) model [29], he qualified it as “a relatively simple approximate method developed for treating problems of the diffraction and stability of shock waves”. The simplicity comes from the fact that the shock front is seen as a surface, $\Gamma(t)$ at time t, evolving under its own normal local speed, D_n, and curvature, independently of the post-shock flow. The shock adjusts itself to changes in the geometry only [30]. As explained by Best [8], Whitham considered the motion of a shock into a uniform gas at rest, down a tube of slowly varying cross sectional area, A, and under some physically grounded hypothesis, he obtained an expression relating the local shock Mach number, M, to A, now known as the A–M relation [32]. Assuming that the shock front is characterized by a level set function ϕ in space Ω, i.e. $\Gamma(t) = \{\mathbf{x} \in \Omega; \phi(t, \mathbf{x}) = 0\}$, one gets the Hamilton–Jacobi equation

$$\partial_t \phi + D_n |\nabla \phi| = 0. \tag{1}$$

For a single pass front, i.e. when each location is reached only once, the shock position can be written as $\phi(t, \mathbf{x}) = \alpha(\mathbf{x}) - c_0 t = 0$ where c_0 is the constant sound speed of undisturbed air, and α is the shock position, also called the travel time by extension. Following Whitham [30], the GSD model reads

$$M(\mathbf{x})\big|\nabla \alpha(\mathbf{x})\big| = 1, \qquad \operatorname{div}\left(\frac{\mathbf{n}(\mathbf{x})}{A(M(\mathbf{x}))}\right) = 0 \tag{2}$$

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where $\mathbf{n} = \frac{\nabla\alpha}{|\nabla\alpha|}$ is the local normal to the front, and $M = D_n/c_0 > 1$ is the local Mach number. This model is hyperbolic provided that $A'(M) < 0$, and can thus develop disturbances on the front which are the traces of waves, not modeled, behind the shock. For GSD, the relation between A and M is

$$\frac{A'(M)}{A(M)} = -\frac{M\lambda(M)}{M^2 - 1} \tag{3}$$

with

$$\begin{cases} 0 < \lambda(M) = \left(1 + \dfrac{2}{\gamma+1}\dfrac{1-\mu^2}{\mu}\right)\left(1 + 2\mu + \dfrac{1}{M^2}\right), \\ 1 \geq \mu = \sqrt{\dfrac{(\gamma-1)M^2 + 2}{2\gamma M^2 - (\gamma - 1)}}, \end{cases} \tag{4}$$

$\gamma > 1$ being the gas polytropic coefficient, μ the local post-shock Mach number and by definition $M > 1$. Thus, the computations only make sense when $M \geq 1$, which is compatible with the eikonal equation. As stated in [30, pp. 296–297], this reservation has to come into consideration when some initial/boundary conditions lead to the apparition of Mach numbers lower than 1 and have to be excluded. The main drawbacks of the model are in the weak shock limit, where the speed of perturbations along the front are underestimated by a factor of two [28] and the absence of regular reflection for shock/solid wall interaction. Nevertheless, in practice, GSD has proven to be fairly accurate for diffraction around a corner, non-regular Mach reflection [30], or accelerating shocks and shown only little deviation for expanding decelerating flows [5]. In the last three decades, Whitham's model has been extended to take into account unsteady flow behind the shock [8–10], non-uniform gases properties [34], and has been applied, among others, to imploding shock waves [11,1], atmospheric propagation [7], detonation in explosives [2,3,6], supersonic engine unstart [28] and astrophysics [14].

This success, linked to the compact model formulation and the dimensional reduction, was supported by the development of three kinds of algorithms. (i) Lagrangian, or front-tracking, methods have first been experimented [15,18]. In such an approach, the shock front is explicitly discretized by markers evolved in time and regularly resampled. This method is natural and quite accurate but difficult to implement in three dimensions, mainly when surface merging or breaking is expected. (ii) Eulerian conservative algorithms [19,20] reduce this difficulty but do not take any advantage of the front locality. Furthermore, they rely on the definition of an a priori propagation direction, not always easy to determine. (iii) Localized level set methods are a good compromise since they handle any kind of surface deformation but in an implicit way. The front shock is obtained from a table of arrival time, also called burn table. A 3D unsteady algorithm, based on the Hamilton–Jacobi form of the GSD system (2) [17], is described in [2–4] for Detonation Shock Dynamics. It compares well with reactive Eulerian model results at a much lower CPU time. Nevertheless, due to the nonlinear nature of GSD equations, nonphysical shocks can form away from the front position and a frequent resampling of the signed distance is mandatory [23].

In this article we propose an alternative approach based on the level set fast-marching paradigm [22], which combines the flexibility of (iii) and the locality of (i) while remaining easy to implement. The algorithm is described in Section 2. Reference solutions for a smooth radial problem and the GSD Riemann problem are derived in Section 3. In Section 4, the comparison to numerical results and the performance of the algorithm are discussed. At last, conclusions are summarized in Section 5.

2. A fast-marching like GSD scheme

In 1988 Osher and Sethian [17] introduced the Eulerian level set method to solve Hamilton–Jacobi equations and the eikonal equation in particular. Unlike the Lagrangian approach, the level set method is simple to implement in 3D, high-order extensions are readily derived and topological properties of the front, as the curvature, are easily calculated. However, assuming a cubic domain in a space of dimension d discretized with N grid points per direction, the unsteady level set method has a complexity $O(N^{d+1})$, which can be quite time consuming when N is large. In the past two decades, several improvements have been proposed in order to reduce this complexity and at same time to enhance the accuracy. As explained in [16], one of the most popular approach is the Fast-Marching Method (FMM) first proposed by Tsitsiklis in [27], rediscovered later by the level set community, and popularized by Sethian [22]. It has since been used with success in a variety of applications. For a single pass front, the complexity of the FMM reduces to $O(N^d \log N)$ and even to $O(N^d)$ under some further assumptions [31].

In this section, we introduce a method to solve the GSD model (2), on a Cartesian grid, based on the fast-marching paradigm. We first reformulate the model as a coupled eikonal-transport system to facilitate its discretization. The numerical method, boundary treatment and implementation details are then given.

2.1. A modified transport equation

As in the work of Besset and Blanc [7] or Aslam [2], the GSD model (2) is rewritten under the local form