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Filtered schemes for Hamilton–Jacobi equations: A simple construction of convergent accurate difference schemes ☆

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ABSTRACT

We build a simple and general class of finite difference schemes for first order Hamilton–Jacobi (HJ) Partial Differential Equations. These filtered schemes are convergent to the unique viscosity solution of the equation. The schemes are accurate: we implement second, third and fourth order accurate schemes in one dimension and second order accurate schemes in two dimensions, indicating how to build higher order ones. They are also explicit, which means they can be solved using the fast sweeping method. The accuracy of the method is validated with computational results for the eikonal equation and other HJ equations in one and two dimensions, using filtered schemes made from standard centered differences, higher order upwinding and ENO interpolation.

1. Introduction

In this work we build a simple and general class of finite difference schemes for first order Hamilton–Jacobi (HJ) Partial Differential Equations. These filtered schemes are almost monotone (in a rigorous sense) and thus provably convergent to the unique viscosity solution of the equation. The schemes are formally accurate: we implement second, third and fourth order accurate schemes in one dimension and second order accurate schemes in two dimensions, indicating how to build higher order ones. They are also explicit, which means they can be solved using the fast sweeping method [26,28], or the fast marching method [22,27] in the case of the eikonal equation.

There are already a large number of discretizations and solvers available for Hamilton–Jacobi equations. Our filtered schemes are designed to remain stable while allowing for a wide choice of accurate discretizations. The simplest approximations are finite difference schemes based on a Cartesian grid. In this class, monotone schemes are provably convergent [6], but only first order accurate [18]. In general, higher order finite difference schemes for HJ equations are neither monotone, nor stable. For example, the centered difference scheme is unstable for the eikonal equation [23, Section 4.3].

Higher order accurate schemes have been built, but only by giving up other desirable properties (e.g. ease of implementation, fast solvers, or the convergence proof). Semi-Lagrangian schemes [11,7], are accurate, but they involve solving the characteristic ordinary differential equations, and are generally more complicated to implement. Central schemes [17]

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achieve second order accuracy, at the expense of a slightly more complicated, non-explicit formulation. The ENO and WENO schemes [19,24,14] are accurate, and while not provably convergent, they are effective in practice. Combinations of WENO and central schemes have been implemented, achieving higher order accuracy [4]. The ENO based schemes use adaptive stencils, which complicates the use of fast solvers (however see [29] for a sweeping method). Fast marching methods require specialized data structures to implement, are usually first order accurate (however see [2] for higher order methods) and only apply to the eikonal equation. A compact upwind second order scheme for the eikonal equation was proposed in [5].

A higher order scheme for Hamilton–Jacobi equations was presented by Abgrall in [1]. Since this scheme uses some ideas similar to ours, we discuss it in further detail in the next paragraphs.

1.1. Contribution of this work

We build filtered schemes by combining a stable, monotone scheme with an accurate (but possibly unstable) scheme. The accurate scheme is not required to be stable on its own: it can simply be standard higher order finite differences, or it can designed to take advantage of known properties of the solutions to the equation under consideration (for example a compact scheme could better avoid singularities in the solution). However, independently of the choice made, the combination of the two schemes is both provably convergent, and (potentially) higher order accurate. We demonstrate that with a judicious choice of the accurate scheme, the higher order accuracy can be achieved. In particular, using one-sided higher order finite differences for the accurate scheme, combined with an upwind monotone scheme results in a very simple, explicit, and accurate scheme for the eikonal equation. We also treat more general cases.

The proof of convergence relies on the classical and well known Barles–Souganidis result [6], which states that monotone, stable, consistent schemes converge. In this paper, convergence of "almost monotone" schemes was mentioned as a remark, but no definition or examples were given. It turns out that filtered schemes, the way we define them, fit very naturally into the framework of the proof, while also being general enough to allow for a variety of schemes. The recent (2009) paper by Abgrall [1] was the first paper to present a provably convergent scheme that blends a monotone scheme with an accurate scheme. The convergence of this scheme also follows from an adaptation of the Barles–Souganidis convergence proof. The small (uniformly bounded) correction to the scheme due to the lack of monotonicity can be absorbed into the term usually seen as the consistency error. The idea of a filtered scheme is then to provide a systematic method to blend a monotone scheme with an accurate scheme and thereby allowing for higher order accuracy. Filtered schemes were previously introduced in [12] in the context of the Monge–Ampère equation. There they were used to overcome the reduction in accuracy based on the wide-stencil monotone scheme. However, the filtered schemes can be applied in a different context to build higher order accurate schemes for the eikonal equation and for more general Hamilton–Jacobi equations.

The schemes we introduce have the following properties

(1) They are simple and easy to implement on Cartesian grids. For example, for the eikonal equation the filtered scheme using the centered difference scheme, is convergent and second order accurate, which results in the simplest second order accurate difference scheme.
(2) Higher order explicit schemes are obtained using higher order upwind interpolation. These higher order schemes can be solved using fast sweeping. If desired, fast marching can be used instead in the case of the eikonal equation.
(3) Other choices of accurate schemes can be used instead: we implement ENO schemes for comparison. Any choice of discretization (e.g. the popular discontinuous Galerkin method) can be used, provided a monotone scheme can also be constructed in the same setting.
(4) For the eikonal equation in one dimension, higher order convergence rates for the numerical solution is proved, even for non-smooth solutions.
(5) For HJ equations (in general), higher order convergence is obtained locally, in regions where the solution is smooth.

1.2. The eikonal equation

We take a particular interest on the eikonal equation

$$\begin{cases} \left|\nabla u(x)\right| = f(x), & \text{for } x \text{ outside } \Gamma, \\ u(x) = g(x), & \text{for } x \text{ on } \Gamma, \end{cases} \tag{1.1}$$

where $f > 0$ and Γ is here a closed, bounded set. The eikonal equation has wide applications in geometric optics, computer vision, optimal control, etc. Moreover, as pointed out in [5], high order schemes are particularly important in the high frequency wave propagation where the eikonal equation is coupled to a transport equation through its gradient [20,25].

1.3. Hamilton–Jacobi equations

We consider HJ equations of the form