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Removal of pseudo-convergence in coplanar and near-coplanar Riemann problems of ideal magnetohydrodynamics solved using finite volume schemes

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ABSTRACT

Numerical schemes for ideal magnetohydrodynamics (MHD) that are based on the standard finite volume method (FVM) exhibit pseudo-convergence in which irregular structures no longer exist only after heavy grid refinement. We describe a method for obtaining solutions for coplanar and near-coplanar cases that consist of only regular structures, independent of grid refinement. The method, referred to as Compound Wave Modification (CWM), involves removing the flux associated with non-regular structures and can be used for simulations in two- and three-dimensions because it does not require explicitly tracking an Alfvén wave. For a near-coplanar case, and for grids with 2^{13} points or less, we find root-square-mean-errors (RMSEs) that are as much as 6 times smaller. For the coplanar case, in which non-regular structures will exist at all levels of grid refinement for standard FVMs, the RMSE is as much as 25 times smaller.

1. Introduction

A Riemann problem is a one-dimensional initial value problem for a conservative system in which a single discontinuity separates two constant states. Riemann problems play an important role in fluid simulations; numerical algorithms in both computational fluid dynamics (CFD) and computational MHD use linear approximations of local Riemann problems for the computation of numerical fluxes [3,6,19].

The ideal MHD equations are more complex than the Euler equations of hydrodynamics. As a result, the number of possible structures is greater for ideal MHD. In addition, the system of equations is non-strictly hyperbolic, which makes non-regular structures such as intermediate shocks and compound waves possible.

Solutions of Riemann problems are composed of multiple structures that emanate away from a discontinuity. A solution is only considered physical if it satisfies entropy and evolutionary conditions [14]. The entropy is $S = p_g/\rho^\gamma$, where p_g is the gas pressure, ρ is the density, and γ is the ratio of specific heats. The entropy condition states that the change in entropy across a shock is zero or larger. The evolutionary condition requires a shock to be structurally stable under small perturbations [14]. In hydrodynamics, the entropy and evolutionary conditions are equivalent.

In the past, intermediate shocks in ideal MHD have been considered unphysical because they are structurally unstable under small perturbations [15]. In recent years, their physicality has been reconsidered. Observations of heliospheric plasma and numerical simulations of bow shocks have provided evidence for their existence. Feng and Wang [11] reported that a

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discontinuity observed by *Voyager* 2 in January 1979 was an intermediate shock. Chao et al. [4] identified an intermediate shock in *Voyager* 1 measurements in 1980. Intermediate shocks have been observed in numerical simulations of bow shocks in both two- and three-dimensions [7,8]. They were first observed in numerical simulations by Brio and Wu [3] whose results have been used extensively as a reference for numerical solutions of the ideal MHD equations. An exact nonlinear solver for ideal MHD that accounts for intermediate shocks was developed by Takahashi and Yamada [22] to investigate non-unique solutions to Riemann problems of ideal MHD. They showed that there were uncountably many non-regular solutions, but only one regular solution to the Riemann problem described by Brio and Wu.

The convergence rates for various implementations of the finite volume method on one-dimensional Riemann problems with non-unique solutions were computed by Torrilhon [24]. All implementations exhibited non-uniform convergence with respect to grid resolution. The schemes produced solutions that converged toward the non-regular solution until a certain level of grid refinement, at which point convergence was to the regular solution. This behavior was referred to as pseudo-convergence, and numerical diffusion was identified as the cause. For the coplanar case, in which the rotation angle is 180°, we argue that convergence to the non-regular solution is expected to always occur, independent of grid resolution, because the transverse velocity and magnetic field are restricted to a single plane.

Because grids with more than 10^4 points are needed to obtain L^1 errors on the order of 10^{-2}, Torrilhon [24] suggested using adaptive mesh refinement (AMR) to reduce the computational costs. AMR can be a powerful computational tool but is complex to implement, and for structured grids, it introduces non-conformity. High order weighted essentially non-oscillatory (WENO) schemes were investigated by Torrilhon and Balsara [26]. The schemes have a spatial accuracy of $2r-1$ in regions where the solution is smooth. They reported results using $r=3$ and $r=5$, giving fifth and ninth order accuracy in space respectively. The higher order schemes converged to the regular solution on coarser grids and were able to obtain L^1 errors on the order of 10^{-2} with half the grid resolution that lower order schemes require, but they still exhibited pseudo-convergence.

We introduce an alternative method for error reduction that does not exhibit pseudo-convergence, Compound Wave Modification (CWM), that requires modifying the flux from the finite volume approximation. The modification is done to the Harden–Lax–van Leer-Discontinuities (HLLD) [18] approximate Riemann solver implemented in the *Athena* MHD code [21,20]. The CWM solutions are compared with one-dimensional exact solutions for a near-coplanar case and the coplanar case. The exact solutions are found using a non-linear Riemann solver that is based on the method described by Dai and Woodward [6] with the rarefaction wave extension by Ryu and Jones [19].

CWM is only suitable for problems of ideal MHD since it is designed to converge to the solution containing regular waves. Evidence suggests non-regular waves are physically admissible for dissipative, non-ideal MHD. Intermediate shocks have been shown to form by wave steepening in the case of dissipative MHD [27,29]. Wu later argued that rotational discontinuities are unstable and will evolve into intermediate shocks in the case of dissipative MHD [28,30]. Time dependent intermediate shocks (TDIS), which do not satisfy the Rankine–Hugoniot, connect two near coplanar states and were also observed in dissipative MHD [31]. Torrilhon [24] compared the behavior of TDIS with pseudo-convergence, in which the regular solution is only obtained after long times in the case of TDIS and for fine grids in the case of pseudo-convergence. Inoue and Inutsuka [12] expanded on earlier finding on the physical admissibility of intermediate shocks in non-ideal MHD to the case of resistive MHD without viscosity. Intermediate shocks may be a physical solution for dissipative MHD therefore the use of CWM would not be appropriate.

The remainder of this paper is organized as follows. In Section 2.1, the MHD equations are presented. In Section 2.2, the classification of the possible discontinuities and shocks of ideal MHD are described. In Section 2.3, an overview of the numerical methods implemented in Riemann solvers is given. In Section 3, Riemann problems with non-unique solutions are introduced and the test cases are described. In Section 4, the CWM method is described and convergence to the correct solution is demonstrated.

2. Ideal MHD and numerical methods

2.1. Ideal MHD

The ideal MHD equations are an approximate description of the interaction between plasma flowing in a region with a magnetic field. They consist of the Euler equations of hydrodynamics and the magnetic induction equation, $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$, for which the divergence-free condition $\nabla \cdot \mathbf{B} = 0$ is satisfied. The effects of resistivity, thermal conductivity, and viscosity are neglected. The equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2.1}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} + \left(p_g + \frac{B^2}{2}\right)\underline{\mathbf{I}} - \mathbf{B} \otimes \mathbf{B}\right] = 0, \tag{2.2}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[\left(E + p_g + \frac{B^2}{2}\right)\mathbf{v} - \mathbf{v} \cdot \mathbf{B} \otimes \mathbf{B}\right] = 0, \quad \text{and} \tag{2.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot [\mathbf{v} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{v}] = 0, \tag{2.4}$$