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An iteration free backward semi-Lagrangian scheme for solving incompressible Navier–Stokes equations ☆

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ABSTRACT

A backward semi-Lagrangian method based on the error correction method is designed to solve incompressible Navier–Stokes equations. The time derivative of the Stokes equation is discretized with the second order backward differentiation formula. For the induced steady Stokes equation, a projection method is used to split it into velocity and pressure. Fourth-order finite differences for partial derivatives are used to the boundary value problems for the velocity and the pressure. Also, finite linear systems for Poisson equations and Helmholtz equations are solved with a matrix-diagonalization technique. For characteristic curves satisfying highly nonlinear self-consistent initial value problems, the departure points are solved with an error correction strategy having a temporal convergence of order two. The constructed algorithm turns out to be completely iteration free. In particular, the suggested algorithm possesses a good behavior of the total energy conservation compared to existing methods. To assess the effectiveness of the method, two-dimensional lid-driven cavity problems with large different Reynolds numbers are solved. The doubly periodic shear layer flows are also used to assess the efficiency of the algorithm.

1. Introduction

The model problems we consider are the incompressible Navier–Stokes equations on a bounded domain $\Omega \subset \mathbb{R}^2$ which are given by

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\Delta\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}|_\Gamma = \mathbf{g} := (g_1, g_2)^T, \end{cases} \tag{1}$$

where $\mathbf{u} = (u, v)^T$, p, ν and Γ denote the velocity field, pressure, kinematic viscosity and the boundary of Ω, respectively. Here, the boundary conditions for velocity fields are considered slip or periodic cases.

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Characteristic-based methods are popular among numerical techniques for solving time-dependent advection-dominated partial differential equations (see [10]). One such method, the backward semi-Lagrangian (BSL) method was designed by Robert [28] to solve meteorological equations in the beginning of the eighties. BSL methods have various significant advantages. (i) They allow a large time step size without damaging the accuracy of the solution. (ii) Unlike pure Lagrangian methods, they do not suffer from mesh-deformation, so that no remeshing is needed. If free boundaries are present, a new mesh should be used at each time step. (iii) They yield algebraic symmetric systems of equations to be solved. Because of these advantages, BSL methods have been extensively used in the numerical simulation of models for fluid dynamics (for examples, see [2–4,9,25,29,34,35] and the references therein).

Most existing BSL methods focus on the development of the interpolation scheme of the solution and spatial discretization schemes. Despite the importance of accuracy in numerical schemes for finding departure points of fluid particles arriving at an Eulerian grid point, it has been received little attention. One reason for this is that the characteristic curves of the particles are described by highly nonlinear ordinary differential equations (ODE) which must be coupled with the solution of the original problems. This is a so-called self-consistency problem and the reason why a high-order time integration scheme is practically hard to implement. It is a known drawback of the BSL methods.

Methods for solving the characteristic equation have a particularly sensitive effect on the accuracy of BSL methods. Traditionally, two main strategies have been proposed and implemented to solve highly nonlinear initial value problems (IVPs) and to find the departure points of the fluid particles. One is an implicit approach requiring iteration [1,32,34]. The other is a substepping method, which is an explicit type [34]. These methods have both second-order convergence accuracy, but it is well known that for a stiff problem, the implicit method achieves a slightly more accurate result compared to the explicit method. Furthermore, when the Reynolds number is large, the implicit method gets a more accurate solution than the explicit method. In addition, the explicit method may work ineffectively in some special cases. The conventional second-order backward integration schemes require an iteration process such as fixed point or Newton iteration when the velocity changes with time (see [23]). At each time and for every spatial point, this iteration process requires the interpolation of solutions which need considerable computational costs. Sometimes, it is prone to accumulate errors during a long-time simulation.

The primary aim of this paper is to develop a BSL method that retains the advantages of conventional second order BSL methods but that does not require the ineffective iteration steps for solving the self-consistent nonlinear problem of the characteristic equations. To accomplish these, we discretize the time derivative of the Stokes equations with the second order backward differentiation formula (BDF2) and apply a projection method in order to split the steady state Stokes equation into velocity and pressure. Secondly, fourth-order finite differences for partial derivatives are used to discretize the Poisson equation for the pressure and the Helmholtz equation for the velocity. Also, the finite discrete linear systems for both the Poisson equation and the Helmholtz equation are solved with a matrix-diagonalization technique. Finally, we apply the error correction techniques, which originated in our recent articles (see [20–22,27]), to solve the highly nonlinear initial value problem of finding the departure points of fluid particles. The error correction method (ECM) is based on the Euler's polygon on each time integration step. To maintain the advantages of the ECM, we suggest a modified Euler's polygon and apply the A-stable midpoint rule for the time integration of the initial value problems. As an interpolation scheme for the solution, the Hermite cubic interpolation technique discussed by Kim et al. [19] is used. The resulting algorithm turns out to be completely iteration free. In particular, it exhibits a good behavior of the total energy conservation compared to existing methods. To assess the effectiveness of the method, two-dimensional lid-driven cavity problems with large different Reynolds numbers are solved. The doubly periodic shear layer flows are also used to assess the efficiency of the proposed method. Throughout these numerical tests, it is shown that the proposed method is quite efficient compared to existing methods.

This paper is organized as follows. In Section 2, we include a brief review of the backward semi-Lagrangian method together with the projection method to deal with steady state Stokes equations. Also, we review the fourth-order finite difference schemes for approximating partial derivatives and the Hermite cubic interpolation theory required for the spatial discretization. Section 3 describes the error correction scheme for solving the self-consistent nonlinear initial value problem for the characteristic curves. In Section 4, we review the matrix-diagonalization technique for solving the finite systems obtained from the discretization based on the finite difference method of the Poisson equation and the Helmholtz equation. Several test problems are performed in Section 5 to exhibit the accuracy and superiority of the proposed method. Finally, in Section 6, a summary for the method, and some discussion of further work are given.

2. Preliminary

The aim of this section is to review a backward semi-Lagrangian scheme for solving the model problem (1) based on the characteristic curve and also to introduce a projection method for a steady Stokes' equation [14,16,24,34]. The splitting scheme is referred to as the rotational form of the velocity-correction scheme in [16] and is also used with semi-Lagrangian schemes in [14,34]. Let $\boldsymbol{\pi}(s,\mathbf{x};t) := (\pi_1(t),\pi_2(t))^T$ be the characteristic curves satisfying the initial value problem given by

$$\begin{cases} \dfrac{d\boldsymbol{\pi}(s,\mathbf{x};t)}{dt} = \mathbf{u}\big(t,\boldsymbol{\pi}(s,\mathbf{x};t)\big), \quad t<s, \\ \boldsymbol{\pi}(s,\mathbf{x};s) = \mathbf{x}, \end{cases} \tag{2}$$