Journal of Computational Physics 282 (2015) 1–22

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Variance reduction through robust design of boundary conditions for stochastic hyperbolic systems of equations

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ARTICLE INFO

Article history:
Received 15 February 2014
Received in revised form 9 September 2014
Accepted 30 October 2014
Available online 6 November 2014

Keywords:
Uncertainty quantification
Hyperbolic system
Initial boundary value problems
Well posed
Stability
Boundary conditions
Stochastic data
Variance reduction
Robust design
Summation by parts

ABSTRACT

We consider a hyperbolic system with uncertainty in the boundary and initial data. Our aim is to show that different boundary conditions give different convergence rates of the variance of the solution. This means that we can with the same knowledge of data get a more or less accurate description of the uncertainty in the solution. A variety of boundary conditions are compared and both analytical and numerical estimates of the variance of the solution are presented. As an application, we study the effect of this technique on Maxwell's equations as well as on a subsonic outflow boundary for the Euler equations.

1. Introduction

In most real-world applications based on partial differential equations, data is not perfectly known, and typically varies in a stochastic way. There are essentially two different techniques to quantify the resulting uncertainty in the solution. Non-intrusive methods [1–7] use multiple runs of existing deterministic codes for a particular statistical input. Standard quadrature techniques, often in combination with sparse grid techniques [8] can be used to obtain the statistics of interest. Intrusive methods [9–16] are based on polynomial chaos expansions leading to systems of equations for the expansion coefficients. This implies that new specific non-deterministic codes must be developed. The statistical properties are obtained by a single run for a larger system of equations. There are also examples of semi-intrusive methods [18,17]. The different procedures are compared in [19,20] and a review is found in [21].

In this paper we take a step back from the technical developments mentioned above and focus on fundamental questions for the governing initial boundary value problem, and in particular on the influence of boundary conditions. Our aim is to minimize the uncertainty or variance of the solution for a given stochastic input. The variance reduction technique in this paper is closely related to well-posedness of the governing initial boundary value problem. In particular it depends on the sharpness of the energy estimate, which in turn depends on the boundary conditions.

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http://dx.doi.org/10.1016/j.jcp.2014.10.061

The technique used in this paper is directly applicable to hyperbolic linear problems such as for example the Maxwell's equations, the elastic wave equations and the linearized Euler equations where the uncertainty is known and limited to the data of the problem. The theoretical derivations are for simplicity and clarity done in one space dimension and for one stochastic variable. The extension to multiple space dimensions and stochastic variables is straightforward and would add more technical details but no principal problems.

We begin by deriving general strongly well posed boundary conditions for our generic hyperbolic problem [22,23,32,33]. These boundary conditions are implemented in a finite difference scheme using summation-by-parts (SBP) operators [24,29–31] and weak boundary conditions [38–41]. Once both the continuous and semi-discrete problems have sharp energy estimates, we turn to the stochastic nature of the problem.

We show how to use the previously derived estimates for the initial boundary value problem in order to bound and reduce the variance in the stochastic problem. Finally we exemplify the theoretical development by numerical calculations where the statistical moments are computed by using the non-intrusive methodology with multiple solves for different values of the stochastic variable [36,37]. The statistical moments are calculated with quadrature formulas based on the probability density distribution [34,35].

The remainder of the paper proceeds as follows. In Section 2 the continuous problem is defined and requirements for well-posedness on the boundary operators for homogeneous and non-homogeneous boundary data are derived. In Section 3 we present the semi-discrete formulation and derive stability conditions. Section 4 presents the stochastic formulation of the problem together with estimates of the variance of the solution. We illustrate and analyze the variance for a model problem in Section 5. In Section 6 we study the implications of this technique on the Maxwell's equations and on a subsonic outflow boundary conditions for the Euler equations. Finally in Section 7 we summarize and draw conclusions.

2. The continuous problem

The hyperbolic system of equations with stochastic data that we consider is,

$$\begin{aligned} u_t + Au_x &= F(x,t,\xi) \quad && 0 \le x \le 1,\ t \ge 0 \\ H_0 u &= g_0(t,\xi) && x = 0,\ t \ge 0 \\ H_1 u &= g_1(t,\xi) && x = 1,\ t \ge 0 \\ u(x,0,\xi) &= f(x,\xi) && 0 \le x \le 1,\ t = 0, \end{aligned} \tag{1}$$

where $u = u(x,t,\xi)$ is the solution, and ξ is the variable describing the stochastic variation of the problem. In general ξ is a vector of multiple stochastic variables, but for the purpose in this paper, one suffice. H_0 and H_1 are boundary operators defined on the boundaries $x = 0$ and $x = 1$. A is a symmetric $M \times M$ matrix which is independent of ξ. $F(x,t,\xi) \in \mathbb{R}^M$, $f(x,\xi) \in \mathbb{R}^M$, $g_0(t,\xi) \in \mathbb{R}^M$ and $g_1(t,\xi) \in \mathbb{R}^M$ are data of the problem.

Remark 1. The limitation to one space and stochastic dimension in (1) is for clarity only. Multiple space and stochastic dimensions add to the technical complexity (for example more complicated quadrature to obtain the statistics of interest), but no principal problems would occur.

In this initial part of the paper, we do not focus on the stochastic part of the problem, that will come later in Section 4. We derive conditions for (1) to be well posed, and focus on the boundary operators H_0 and H_1.

2.1. Well-posedness

Letting $F = 0$, we multiply (1) by u^T and integrate in space. By rearranging and defining $\|u\|^2 = \int_\Omega u^T u dx$ we get,

$$\|u\|_t^2 = \left(u^T A u\right)_{x=0} - \left(u^T A u\right)_{x=1}. \tag{2}$$

Due to the fact that A is symmetric, we have

$$A = X \Lambda X^T, \qquad X = [X_+, X_-], \qquad \Lambda = \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix}. \tag{3}$$

In (3), X_+ and X_- are the eigenvectors related to the positive and negative eigenvalues respectively. The eigenvalue matrix Λ is divided into diagonal block matrices Λ^+ and Λ^- containing the positive and negative eigenvalues respectively. Using (2) and (3) we get,

$$\|u\|_t^2 = \left(X^T u\right)_0^T \Lambda \left(X^T u\right)_0 - \left(X^T u\right)_1^T \Lambda \left(X^T u\right)_1. \tag{4}$$

The boundary conditions we consider are of the form

$$\begin{aligned} H_0 u &= \left(X_+^T - R_0 X_-^T\right) u_0 = g_0, \quad x = 0 \\ H_1 u &= \left(X_-^T - R_1 X_+^T\right) u_1 = g_1, \quad x = 1, \end{aligned} \tag{5}$$