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Hermite WENO limiting for multi-moment finite-volume methods using the ADER-DT time discretization for 1-D systems of conservation laws

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ABSTRACT

New Hermite Weighted Essentially Non-Oscillatory (HWENO) interpolants are developed and investigated within the Multi-Moment Finite-Volume (MMFV) formulation using the ADER-DT time discretization. Whereas traditional WENO methods interpolate pointwise, function-based WENO methods explicitly form a non-oscillatory, high-order polynomial over the cell in question. This study chooses a function-based approach and details how fast convergence to optimal weights for smooth flow is ensured. Methods of sixth-, eighth-, and tenth-order accuracy are developed. These are compared against traditional single-moment WENO methods of fifth-, seventh-, ninth-, and eleventh-order accuracy to compare against more familiar methods from literature. The new HWENO methods improve upon existing HWENO methods (1) by giving a better resolution of unreinforced contact discontinuities and (2) by only needing a single HWENO polynomial to update both the cell mean value and cell mean derivative.

Test cases to validate and assess these methods include 1-D linear transport, the 1-D inviscid Burger's equation, and the 1-D inviscid Euler equations. Smooth and non-smooth flows are used for evaluation. These HWENO methods performed better than comparable literature-standard WENO methods for all regimes of discontinuity and smoothness in all tests herein. They exhibit improved optimal accuracy due to the use of derivatives, and they collapse to solutions similar to typical WENO methods when limiting is required. The study concludes that the new HWENO methods are robust and effective when used in the ADER-DT MMFV framework. These results are intended to demonstrate capability rather than exhaust all possible implementations.

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1. Introduction

Systems of conservation laws govern the evolution of many important physical phenomena, and they form the core of many simulation codes. An algorithm that numerically solves a non-linear system of conservation laws must balance multiple constraints, including accuracy, robustness, runtime, and parallel efficiency. Weighted Essentially Non-Oscillatory (WENO) methods [1,2] are good candidates in balancing these constraints because they simultaneously allow an optimal accuracy over a given stencil of cells, while providing robustness only where needed. WENO methods provide requisite damping where needed without additional parallel data transfers, which is not generally true for hyperdiffusive filters and

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Flux Corrected Transport. ADER time discretizations also balance these constraints well, as they require no quadrature in the spatial discretization and no steps/stages in the time discretization. Combining ADER and WENO is a natural choice because only one WENO limiting is needed per large time step, and the overhead of WENO versus traditional reconstruction is relatively smaller with ADER.

Hermite WENO (HWENO) methods [3–6] use not only a stencil of values but also derivatives to reconstruct, providing a more accurate optimal reconstruction than standard WENO methods. Also, HWENO methods can perform multi-moment simulation at significantly larger time steps than Galerkin methods, and they are much more easily limited than Galerkin methods. In fact, the maximum stable CFL of HWENO ADER methods in 1-D is unity, no matter what the order of accuracy is. The center cell's derivative comes at no cost in terms of parallel data transfer, which makes HWENO methods relatively more parallel efficient than WENO methods. Thus, ADER with HWENO seems like a natural choice in balancing the many constraints of efficient numerical integration of conservation law systems.

However, HWENO methods have suffered some difficulties in previous derivations. In [4], one must compute two separate HWENO interpolants in order to evolve the scheme, which is prohibitively expensive to be efficient in practice. The HWENO methods of this study require only one HWENO interpolant to be formed, and it updates both the cell-averaged value and cell-averaged derivative of the state. In [3], there is the problem that the method degrades to the first-order-accurate Godunov method, spatially, when discontinuities become sufficiently steep. This is not so much of a problem for reinforced discontinuities such as shocks, but for contact discontinuities and linear transport discontinuities, this will excessively diffuse discontinuities. The HWENO methods of this study perform more accurately than a first-order-accurate Godunov at contact discontinuities, mimicking the behavior of more traditional WENO schemes. Finally, the HWENO methods herein are more similar to [3] than to [2] in the sense that they directly form full polynomials rather than point values. This provides remarkable flexibility in terms of deriving a new method, and that flexibility is demonstrated in the new methods.

First, the MMFV method and the ADER-DT time discretization will be described in detail in Section 2. Then, the design philosophy of the new HWENO methods is detailed in Section 3, and four HWENO methods are derived at sixth-, eighth-, and tenth-order accuracies with varying specification of low-ordered polynomials. In Section 4, numerical experiments are performed to validate and assess the new HWENO methods against a baseline WENO method that is common from literature. Finally, conclusions are drawn in Section 5.

2. Multi-moment finite-volume, ADER-DT framework

2.1. Multi-moment finite-volume evolution

For the purposes of investigating new Hermite WENO reconstructions for multi-moment finite-volume methods using the ADER time discretization, this study is concerned with a generic 1-D system of conservation laws of the form:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \mathbf{0} \tag{1}$$

where $\mathbf{q}$ and $\mathbf{f}$ are vector quantities containing the conserved variables and fluxes, respectively. For the first evolution, (1) is integrated over a space–time block defined by $\Omega_i \times [t_n, t_{n+1}]$, where $\Omega_i \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $x_{i\pm\frac{1}{2}} = x_i \pm \Delta x/2$, $t_{n+1} = t_n + \Delta t$. This gives rise to:

$$\begin{aligned}
&\bar{\mathbf{q}}_{i,n+1} - \bar{\mathbf{q}}_{i,n} + \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+\frac{1}{2},n} - \mathcal{F}_{i-\frac{1}{2},n}) = \mathbf{0} \\
&\mathcal{F}_{i-\frac{1}{2},n} = \mathcal{R}\left(\widehat{\mathbf{q}}_{i,n}(x_{i-\frac{1}{2}}), \widehat{\mathbf{q}}_{i+1,n}(x_{i-\frac{1}{2}}), \widehat{\mathbf{f}}_{i,n}(x_{i-\frac{1}{2}}), \widehat{\mathbf{f}}_{i+1,n}(x_{i-\frac{1}{2}})\right) \\
&\bar{\mathbf{q}}_{i,n} = \frac{1}{\Delta x}\int_{\Omega_i} \mathbf{q}_{i,n}(x, t_n) d\Omega \\
&\widehat{\mathbf{f}}_{i,n}(x) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{f}_{i,n}(x, t) dt \\
&\widehat{\mathbf{q}}_{i,n}(x) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{q}_{i,n}(x, t) dt
\end{aligned} \tag{2}$$

where $\mathcal{R}$ is a flux function intended to reconcile different time-averaged fluxes sampled at the same cell interface. Note that the flux function is performed on time-averaged states and fluxes, and therefore, it should be a linear function rather than a non-linear function.

For the second evolution equations, intended to evolve the cell-averaged first-derivative, (1) is first differentiated in space and then integrated over $\Omega_i \times [t_n, t_{n+1}]$. This gives rise to: