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An axisymmetric multiple-relaxation-time lattice Boltzmann scheme

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ABSTRACT

A multiple-relaxation-time (MRT) lattice Boltzmann (LB) scheme developed for axisymmetric flows recovers the complete continuity and Navier–Stokes equations. This scheme follows the strategy of the standard D2Q9 model by using a single particle distribution function and a simple "collision-streaming" updating rule. The extra terms related to axisymmetry in the macroscopic equations are recovered by adding source terms into the LB equation, which are simple and involve no gradients. The compressible effect retained in the Navier–Stokes equations is recovered by introducing a term related to the reversed transformation matrix for MRT collision operator, so as to produce a correct bulk viscosity, making it suitable for compressible flows with high frequency and low Mach number. The validity of the scheme is demonstrated by testing the Hagen–Poiseuille flow and 3D Womersley flow, as well as the standing acoustic waves in a closed cylindrical chamber. The numerical experiments show desirable stability at low viscosities, enabling to simulate a standing ultrasound field in centimeters space.

1. Introduction

In the last two decades, the lattice Boltzmann (LB) method has developed rapidly into a powerful tool for numerical study of complex fluid flows [1–3]. As an alternative to solve the partial differential equations, the LB method is characterized by the local formulation of the particle interactions, simple updating rule on grids, intrinsic adaptability to parallel computing, and easy implementation of the boundary conditions and interfacial phenomena. Though originally devised for the Navier–Stokes equations [4–6], it has also been extended to broader physical areas beyond the fluid dynamics. These include the acoustics [7,8], electrodynamics [9,10], quantum mechanics [11,12], diffusion [13,14], phase separation [15,16], phase transition [17,18], and so on.

Axisymmetric flow problems frequently appear in scientific research and practical applications. Making use of the axisymmetry, the computational efficiency can be significantly enhanced by reducing the dimensions of the coordinate system. In recent years, many efforts have been paid to develop the axisymmetric LB models on the two-dimensional (2D) square lattice. Halliday et al. [19] firstly proposed a model by adding "source" or "force" terms into the standard 2D Cartesian lattice Boltzmann equation, so as to recover the cylindrical polar coordinate form of the continuity and Navier–Stokes equations. Following this idea, several modified axisymmetric LB models were offered [20–22], and developed even for thermal flows [23] and multiphase flows [24–26]. Recently, Li et al. [27] and Zhou [28] proposed improved axisymmetric LB models that contain fewer source terms and are free of velocity gradients, so that the algorithm of the collision step

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is effectively simplified. Axisymmetric LB methods other than Halliday's methodology has also been developed based on the vorticity-stream-function equations [29,30] or derived from the continuous Boltzmann equation in cylindrical coordinates [31]. Nevertheless, the advantages of Halliday's methodology are obvious since it inherits the general benefits of the standard LB method and is easy to treat the boundary conditions.

The traditional lattice Boltzmann equation usually uses the Bhathagar–Gross–Krook (BGK) collision operator. As an extension to the single relaxation time used in the BGK model, the multiple-relaxation-time (MRT) collision scheme was developed [32–35], which separates various hydrodynamic modes and allows them to relax at different time scales. The MRT-LB method includes more physics and shows better numerical stability, especially at low fluid viscosities. In the modeling of the axisymmetric lattice Boltzmann equations, the MRT collision algorithm was also taken into consideration [25,27,36,37]. However, these existing axisymmetric MRT models were devised only for incompressible flows, and the additional relaxation parameters were employed mainly to promote the computing stability rather than modeling the realistic physical properties of the fluids. For example, the bulk viscosity was usually over-evaluated by setting a small relaxation rate for the energy mode. In this paper, an axisymmetric MRT LB model is proposed in the frame of the 2D standard lattice, which aims to recover the complete continuity and Navier–Stokes equations at the limit of low Mach number. This model offers proper compressible effect and reasonable bulk viscosity, and can be applied to simulate fluid flows with considerable compressibility and finite velocity amplitude, such as high frequency acoustic waves in cylindrical symmetry. It also inherits the benefits of the standard LB scheme by using a single distribution function and a simple up-dating rule and is free from gradient terms.

The rest of the paper is organized as follows. In Section 2, the axisymmetric continuity and Navier–Stokes equations are described. In Section 3, the axisymmetric MRT LB model is presented, and then subjected to the Chapman–Enskog analysis and explicitation. The method is numerically validated In Section 4 and finally concluded in Section 5.

2. Axisymmetric flow equations with compressibility

The complete continuity and Navier–Stokes equations for the axisymmetric flows in cylindrical coordinates can be written as follows [38]:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u_i)}{\partial x_i}=-\frac{\rho u_r}{r}, \tag{1}$$

$$\begin{aligned}\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_j)}{\partial x_j}=&-\frac{\partial p}{\partial x_i}-\frac{\rho u_r u_i}{r}+\frac{\partial}{\partial x_j}\left[\eta\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]+\frac{1}{r}\left[\eta\left(\frac{\partial u_r}{\partial x_i}+\frac{\partial u_i}{\partial r}\right)\right]-2\eta\frac{u_r\delta_{ir}}{r^2}\\&+\frac{\partial}{\partial x_i}\left[\eta'\left(\frac{\partial u_k}{\partial x_k}+\frac{u_r}{r}\right)\right],\end{aligned} \tag{2}$$

where ρ is the density, p is the pressure, t is the time, i, j and k are the indexes standing for r or z, i.e., the coordinates in radial and axial directions, u_i is the component of macroscopic velocity in i direction, η and η' are the shear viscosity and the second viscosity, respectively, δ_{ij} is the Kronecker delta function, and the repeated indexes indicate the Einstein summation convention, which means a summation over the space coordinates.

The last term in Eq. (2) vanishes if the fluid is assumed to be incompressible. In order to include the compressible effect and reformulate the lattice Boltzmann equation in a reasonable way, this term is retained here and will be used in Section 3.

3. Axisymmetric MRT lattice Boltzmann model

3.1. Lattice Boltzmann equation

The present multiple-relaxation-time lattice Boltzmann equation is based on the standard two-dimensional nine-velocity (D2Q9) lattice [5] by employing a substitution of the coordinates, $(x, y) \to (r, z)$. The discrete microscopic velocity $\mathbf{e}_\alpha = (e_{\alpha r}, e_{\alpha z})$ of a particle in the α link is written as:

$$\mathbf{e}_\alpha=\begin{cases}(0,0), & \alpha=0,\\ e(\cos\frac{(\alpha-1)\pi}{2},\sin\frac{(\alpha-1)\pi}{2}), & \alpha=1,2,3,4,\\ \sqrt{2}e(\cos\frac{2(\alpha-5)\pi+\pi}{4},\sin\frac{2(\alpha-5)\pi+\pi}{4}), & \alpha=5,6,7,8,\end{cases} \tag{3}$$

and the corresponding local equilibrium distribution function f_α^{eq} is defined as

$$f_\alpha^{\text{eq}}=w_\alpha\rho\left[1+3\frac{\mathbf{e}_\alpha\cdot\mathbf{u}}{e^2}+\frac{9}{2}\frac{(\mathbf{e}_\alpha\cdot\mathbf{u})^2}{e^4}-\frac{3}{2}\frac{\mathbf{u}\cdot\mathbf{u}}{e^2}\right], \tag{4}$$

where e is the unit velocity, and w_α is the weight factor given by