Journal of Computational Physics 281 (2015) 148–176

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

High-order central Hermite WENO schemes on staggered meshes for hyperbolic conservation laws ☆

Zhanjing Tao [a], Fengyan Li [b], Jianxian Qiu [a,*]

[a] School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling and High-Performance Scientific Computation, Xiamen University, Xiamen, Fujian 361005, PR China

[b] Department of Mathematical Science, Rensselaer Polytechnic Institute, 110 8th Street, Troy, NY 12180, United States

ARTICLE INFO

Article history:
Received 5 August 2014
Received in revised form 11 October 2014
Accepted 13 October 2014
Available online 22 October 2014

Keywords:
Finite volume method
Central schemes
Hermite WENO
Lax–Wendroff
Natural continuous extension (NCE) of Runge–Kutta
High-order accuracy

ABSTRACT

In this paper, we propose a class of high-order schemes for solving one- and two-dimensional hyperbolic conservation laws. The methods are formulated in a central finite volume framework on staggered meshes, and they involve Hermite WENO (HWENO) reconstructions in space, and Lax–Wendroff type discretizations or the natural continuous extension of Runge–Kutta methods in time. Compared with central WENO methods, the spatial reconstruction used here is much more compact; and unlike the original HWENO methods, our proposed schemes require neither flux splitting nor the use of numerical fluxes. In the system case, local characteristic decomposition is applied in the reconstructions of cell averages to enhance the non-oscillatory property of the methods. The high resolution and robustness of the methods in capturing smooth and non-smooth solutions are demonstrated through a collection of one- and two-dimensional scalar and system of examples.

1. Introduction

Hyperbolic conservation laws arise in a wide range of applications in science and engineering, such as aerodynamics, meteorology and weather prediction, astrophysical modeling, multi-phase flow problems, and the study of explosion and blast waves. In general the exact solutions of such equations are not available, and they can also develop discontinuous features, e.g. shocks, compound waves, etc., regardless of the smoothness of the initial and boundary data. It has been an active and important research area to design accurate and robust methods for numerically simulating hyperbolic conservation laws.

In this paper, we design high-order central Hermite WENO (weighted essentially non-oscillatory, C-HWENO) schemes for solving one- and two-dimensional hyperbolic conservation laws

$$\begin{cases} u_t + \nabla \cdot f(u) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

☆ The research was partially supported by NSFC grants 91230110 and 11328104, NSF grants DMS-0847241 and DMS-1318409, and the Fundamental Research Funds for Xiamen University, Grant No. 201412G005.

* Corresponding author.
E-mail addresses: tzjnchy555@163.com (Z. Tao), lif@rpi.edu (F. Li), jxqiu@xmu.edu.cn (J. Qiu).

http://dx.doi.org/10.1016/j.jcp.2014.10.027
0021-9991/

with suitable initial and boundary conditions. Here (1.1) can be scalar or a system, and it is often nonlinear. Our methods use Hermite WENO (HWENO) reconstructions as spatial discretizations, and Lax–Wendroff type discretizations or the natural continuous extension of Runge–Kutta methods as time discretizations, in a central finite volume formulation on staggered meshes. Compared with central WENO (C-WENO) schemes, one major advantage of C-HWENO schemes is the compactness in the spatial reconstruction. Compared with the original HWENO schemes, the proposed methods require neither flux splitting nor the use of numerical fluxes that are often exact or approximate Riemann solvers. When (1.1) is a system, local characteristic decomposition is applied in the reconstruction of cell averages to enhance non-oscillatory property of the schemes.

WENO schemes are high-order finite volume or finite difference methods widely used for hyperbolic conservation laws, with attractive property of maintaining both uniform high-order accuracy and an essentially non-oscillatory shock transition. They were designed based on the successful ENO (essentially non-oscillatory) schemes [7,29,30], and improve in robustness, smoothness of fluxes, better steady state convergence, accuracy in smooth region of the solutions, and efficiency. The first WENO scheme was constructed in [18] as a third-order finite volume method in one space dimension. In [9], third and fifth-order finite difference WENO schemes in multiple dimensions were constructed, with a general framework to design smoothness indicators and nonlinear weights. Finite difference WENO schemes of higher order accuracy (i.e. seventh- to eleventh-order) were proposed in [2], while the finite volume versions on structured and unstructured meshes were investigated in, e.g. [5,8,14,27,23]. In [27], a simple and effective technique for handling negative linear weights without a need to get rid of them was proposed, and this technique is also adopted in this paper. We refer to [28] for a detailed review of WENO schemes. It is known that higher order accuracy in a finite difference or finite volume framework relies on enlarging the stencil for reconstructions. To improve the compactness while keeping the accurate and non-oscillatory properties of the methods, a fifth-order finite volume HWENO scheme was proposed in [24] for one dimension, with a fourth-order one in [26,34] for two dimensions. The compactness of HWENO methods is achieved by evolving not only the solution but also its first spatial derivative(s), and they are both used in the high-order spatial reconstructions. Some other related earlier work includes [31,4,21].

Our proposed methods are also related to central schemes, which can be regarded as extensions of the classical Lax–Friedrichs method [6]. A second-order central scheme was first developed in [22] by Nessyahu and Tadmor, and it requires neither numerical fluxes, that are exact or approximate Riemann solvers, nor flux splitting. For the system case, numerical tests show that local characteristic decomposition is also not needed. Motivated by the simplicity and robustness of the second-order central scheme, various high-order or semi-discrete versions as well as extensions to multiple dimensions were explored in [19,3,1,10–12,14,15,23]. In a series of recent papers, the ENO and WENO reconstruction techniques have been successfully integrated into the central framework. A one-dimensional central ENO (C-ENO) scheme was introduced in [3]. The third- and fourth-order C-WENO schemes were developed in [14–17] for one- and two-dimensional conservation laws. In [23], fifth- and ninth-order C-WENO schemes were constructed based on finite volume formulation on staggered meshes, and they used the natural continuous extension of Runge–Kutta methods in time. Numerical experiments in [23] also demonstrate that the local characteristic decomposition is still necessary to control spurious oscillations when the order of accuracy is high, for both the central WENO schemes on staggered meshes and the upwind WENO schemes on non-staggered meshes.

When upwind type WENO schemes are used as spatial discretizations for solving hyperbolic conservation laws, they are often combined with explicit nonlinearly stable TVD Runge–Kutta time discretizations [29] following method of lines approaches. The schemes developed in the present work, on the other hand, are defined on staggered meshes, and they are more of fully discrete schemes themselves. For such methods, one can no longer directly apply the explicit nonlinearly stable TVD Runge–Kutta methods to achieve high-order accuracy in time. Instead, we choose to use two other time discretizations: the Lax–Wendroff type discretizations, and the natural continuous extension of Runge–Kutta methods. The one-step one-stage Lax–Wendroff type time discretization, which is also called the Taylor type, is based on the idea of the classical Lax–Wendroff scheme [13]. It relies on converting the time derivatives in a temporal Taylor expansion of the solution into spatial derivatives by repeatedly using the governing equations and their differentiated forms. The original finite volume ENO schemes in [7] used this approach for the time discretization. In [25], a Lax–Wendroff time discretization procedure was developed for high-order finite difference WENO schemes. In contrast to upwind schemes, many fully discrete high-order central schemes [3,14–17,23] use Runge–Kutta methods with the aid of the natural continuous extension [33] as time discretizations. The natural continuous extension of Runge–Kutta technique is based on standard Runge–Kutta time discretizations, and it provides approximations of comparable accuracy for the solutions at intermediate time over each time step. It works well with spatial discretizations defined on staggered meshes yet does not require much additional cost than standard Runge–Kutta methods.

The organization of this paper is as follows. In Section 2, we describe in detail the construction and implementation of C-HWENO schemes with Lax–Wendroff type time discretizations for one- and two-dimensional scalar and system equation (1.1). In Section 3, we present the construction and implementation of C-HWENO schemes with the natural continuous extension of Runge–Kutta methods as time discretizations, and the HWENO reconstructions here are similar to those in Section 2. In Section 4, extensive numerical examples are provided to demonstrate the performance of the proposed schemes for smooth and non-smooth examples in one and two dimensions. Concluding remarks are made in Section 5.