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High order parametrized maximum-principle-preserving and positivity-preserving WENO schemes on unstructured meshes

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A B S T R A C T

In this paper, we generalize the maximum-principle-preserving (MPP) flux limiting technique developed by Xu (2013) [20] to a class of high order finite volume weighted essentially non-oscillatory (WENO) schemes for scalar conservation laws and the compressible Euler system on unstructured meshes in one and two dimensions. The key idea of this parameterized limiting technique is to limit the high order numerical flux with a first order flux which preserves the MPP or positivity-preserving (PP) property. The main purpose of this paper is to investigate the flux limiting approach with high order finite volume method on unstructured meshes which are often needed for solving some important problems on irregular domains. Truncation error analysis based on one-dimensional nonuniform meshes is presented to justify that the proposed MPP schemes can maintain third order accuracy in space and time. We also demonstrate through smooth test problems that the proposed third order MPP/PP WENO schemes coupled with a third order Runge–Kutta (RK) method attain the desired order of accuracy. Several test problems containing strong shocks and complex domain geometries are also presented to assess the performance of the schemes.

1. Introduction

In this paper, we are interested in the scalar conservation law:

$$u_t + \nabla \cdot \mathbf{F}(u) = 0 \tag{1.1}$$

and the compressible Euler system:

$$\xi_t + f(\xi)_x + g(\xi)_y = 0, \tag{1.2}$$

with

$$\xi = \begin{pmatrix} \rho \\ m_u \\ m_v \\ E \end{pmatrix}, \qquad f(\xi) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix}, \qquad g(\xi) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix},$$

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and

$$m_u = \rho u, \qquad m_v = \rho v, \qquad E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho\left(u^2 + v^2\right)$$

where ρ is the density, $(u, v)^T$ is the velocity, m_u and m_v are the momentums, p is the pressure, E is the total energy and γ is the specific heat ratio.

Many numerical methods have been developed for solving (1.1) and (1.2) over the recent decades, such as the discontinuous Galerkin (DG) method [2], the finite volume/finite difference essentially non-oscillatory (ENO) schemes [6,10], and finite volume/finite difference WENO schemes [10,12]. Among various methods, WENO schemes are shown to be very robust and efficient especially when solutions may contain discontinuities, sharp gradient regions and other complicated solution structures. Moreover, finite volume WENO schemes have more flexibility in terms of mesh structure compared to finite difference WENO schemes. In particular, finite volume WENO schemes have been applied to unstructured meshes with arbitrary partition for complex domain geometries [3,7,13,29], while finite difference WENO schemes can only be applied to a uniform or smoothly varying mapped grid. In this paper, we focus our discussion on finite volume WENO schemes.

The solution to the scalar conservation law (1.1) has MPP property such that, if the initial condition is bounded $u_m \leq u(\mathbf{x}, t_0) \leq u_M$, then $u_m \leq u(\mathbf{x}, t) \leq u_M$ for all the future times $t > t_0$. Similarly, the solutions to the compressible Euler system have a PP property such that both densities and pressures must maintain positive in every situation. However, the existing high order schemes for solving (1.1) and (1.2) do not necessarily retain the MPP/PP property in the numerical solutions. In particular, if the solution to (1.2) contains low density or pressure, high order schemes might produce negative density or pressure, leading to an ill-posed problem that typically causes failure of the numerical algorithm. The situation was restively stagnant until the recent work by Zhang and Shu [23,24]. In this work, arbitrary high order finite volume WENO schemes and DG methods are developed to preserve MPP and PP properties, by limiting the reconstructed polynomials around cell averages. Following a similar idea, Zhang and Shu extended their schemes to two-dimensional unstructured meshes [25,27]. They further demonstrated that the schemes were able to the maintain designed order of accuracy under a CFL constraint. Afterwards, Hu et al. [9] also developed a flux cut-off limiter applied to finite difference WENO schemes which maintains the PP property for compressible Euler system. More recently, a parametrized MPP flux limiter is developed to maintain the MPP property for scalar hyperbolic conservation laws [11,20]. The main advantage of this new parametrized limiter is that the designed order of accuracy of the base WENO/DG schemes are maintained without excessively restricting the CFL. Later in [19], Xiong et al. improved the CFL constraint and reduced computational cost by applying the parametrized MPP flux limiter to the final RK stage only. It was also proven in [19] that the parametrized MPP flux limiter can maintain up to third order accuracy in space and time for one-dimensional nonlinear scalar conservation laws on uniform meshes. This limiter is also extended to high order PP finite difference WENO schemes for the compressible Euler system in [18] and high order MPP finite volume WENO methods for scalar convection-dominated problems on uniform meshes [21].

In this paper, we will generalize the parametrized flux limiter to finite volume WENO schemes on unstructured meshes, and perform numerical experiments for both scalar equations and the Euler system of compressible gas dynamics. To accompany the numerical results, error analysis on one-dimensional nonuniform meshes is provided. There are two main types of WENO reconstruction in the literature. In the first type of reconstruction, the order of WENO schemes is not higher than the degree of the reconstructed polynomials on each small stencil. i.e., the nonlinear WENO weights are only designed for the purpose of stability, see [3–5,15,16] for details. The second type consists of WENO schemes whose order of accuracy is higher than the degree of the reconstructed polynomials on each small stencil, see, for example, [7,28,29]. Compared with the first type of WENO schemes, the second type of WENO schemes are more difficult to construct but have a more compact stencil of the same accuracy. Based on those two different approaches, a hybrid approach was also proposed in [13] to deal with distorted local mesh geometries. In the implementation of this work, we will use the second type of WENO reconstruction and the hybrid approach. We refer them as WENO-C and WENO-H respectively in the following sections.

The rest of the paper is organized as follows. In Section 2, we will briefly review finite volume schemes for scalar hyperbolic conservation laws and the parametrized MPP flux limiter. In Section 3, we will provide error analysis on one-dimensional nonuniform meshes to show that the MPP schemes can preserve high order accuracy without any sacrifice on CFL constraints. In Section 4, we will present high order MPP and PP finite volume WENO schemes on two-dimensional unstructured meshes for scalar conservation laws and compressible Euler system. Numerical examples of the third order finite volume WENO schemes for smooth problems and some problems containing strong shocks and complex geometries will be shown in Section 5. The conclusion is given in Section 6.

2. One-dimensional finite volume MPP flux limiter for scalar equations

In this section, we first present the general framework of an MPP limiter applied to a finite volume scheme on a nonuniform mesh.

We consider a one-dimensional scalar conservation law:

$$\begin{cases} u_t + f(u)_x = 0, \quad x \in [0, 1]; \\ u(x, 0) = u_0(x), \end{cases} \tag{2.1}$$