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Robust high-order space–time conservative schemes for solving conservation laws on hybrid meshes

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ABSTRACT

In this paper, the second-order space–time conservation element and solution element (CE/SE) method proposed by Chang (1995) [3] is implemented on hybrid meshes for solving conservation laws. In addition, the present scheme has been extended to high-order versions including third and fourth order. Most methodologies of proposed schemes are consistent with that of the original CE/SE method, including: (i) a unified treatment of space and time (thereby ensuring good conservation in both space and time); (ii) a highly compact node stencil (the solution node is calculated using only the neighboring mesh nodes) regardless of the order of accuracy at the cost of storing all derivatives. A staggered time marching strategy is adopted and the solutions are updated alternatively between cell centers and vertexes. To construct explicit high-order schemes, second- and third-order derivatives are calculated by a modified finite-difference/weighted-average procedure which is different from that used to calculate the first-order derivatives. The present schemes can be implemented on a wide variety of meshes, including triangular, quadrilateral and hybrid (consisting of both triangular and quadrilateral elements). Beyond that, it can be easily extended to arbitrary-order schemes and arbitrary shape of polygonal elements by using the present methodologies. A series of common benchmark examples are used to confirm the accuracy and robustness of the proposed schemes.

1. Introduction

In performing the direct numerical simulation (DNS) and large eddy simulation (LES) of turbulent flows and computational aeroacoustics (CAA), treating complex configurations in an accurate and efficient manner is essential. Under normal circumstances, structured quadrilateral meshes have better computational accuracy than unstructured triangular meshes. However, multi-block meshes are required to treat complex configurations. Although triangular meshing schemes have a good ability to model complex configurations, they have difficulties in generating a viscous mesh. Thus, in solving actual engineering and scientific problems, hybrid meshes combining triangular and quadrilateral meshes are generally required. On the choice between high-order schemes and typical second-order schemes, previous research has shown that the former

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usually have significant potentials to achieve higher efficiency in terms of number of grids required to accurately resolve some important flow features than the latter [1]. As a result, the problem of developing high-order numerical methods implemented on hybrid meshes has attracted increasing interest in the computational fluid dynamics field in recent years.

When constructing a finite volume/difference method on a quadrilateral or a triangular mesh, one is given the option of choosing the upwind scheme or the centered scheme, each with its own advantages and drawbacks [2]. The second-order space–time Conservation Element and Solution Element (CE/SE) method, developed by Chang and coworkers [3–13], is a centered scheme based on a non-dissipative core. Compared with the N–T scheme (Nessyahu and Tadmor, 1990) [14], the mesh, the balancing of fluxes, and the updates of the cell average quantities are essentially identical. The difference is in the estimation of the slopes (of the linear interpolants, see [2]). In the "a–ϵ" CE/SE scheme, the parameter ϵ is involved in the calculation of slopes and is used to control the numerical dissipation. When $\epsilon = 0$, the scheme is reversible in time and dissipation-free but cannot be directly used to capture shock phenomena. When $\epsilon = 1/2$ (employed here and in most practical calculations of CE/SE), the CE/SE scheme is not reversible anymore, and it has the (not-necessarily-desirable) property that when time step is of size zero, the solution changes with time due to staggering (after two half time steps) and does not remain the same as the initial data.

The original CE/SE schemes were constructed on triangular elements in 2D [4] and tetrahedrons in 3D [13]. However, later two different extensions of the CE/SE scheme were based on quadrilateral elements in 2D [11,15] and hexahedral elements in 3D [11,16]. Generally speaking, the above CE/SE schemes can be categorized as non-staggered and staggered versions. Methodologies used in the two versions are essentially the same, except the time marching strategy. In the non-staggered version [11], the solution elements (SEs) and conservation elements (CEs) are twice as big as the underlying cells, and the collection of all SEs (or CEs) will cover the computational domain twice over; the solution is updated at all cell centers (and does not zigzag). However, the staggered version updates the solution alternatively between any two of the three sets of data: at cell vertexes, at cell centers, and at midpoints of edges. In present study, a staggered version is extended to hybrid meshes. At time n, the solution is stored at the cell vertexes, i.e., centers of the staggered cells; at time $n + 1/2$, the solution is updated and stored at original cell centers; at time $n + 1$, the solution is back at the original cell vertexes.

Although the original CE/SE scheme is only a second-order scheme, its resolution of strong discontinuities is comparable with that of high-order methods such as the fourth-order Essential Non-Oscillatory scheme [5]. It has already been widely applied in simulations of sound wave propagation [9], aero-acoustics [5,17], steady viscous flows [18], supersonic capsule flows [19], hypersonic viscous flows [20], elastic wave propagation in solids [21], ultra-relativistic Euler equation problems [22], magneto-hydrodynamic flows [10,23–27], chemical reactive flows [12,16,28–30], multi-material elastic–plastic flows [15], spall fracture phenomena [31] and many electrical engineering problems [32]. However, in the following paper, we will see that the computational accuracy of the CE/SE scheme is not as satisfactory as high order schemes, when applied to certain complex fine fluid structures. As a result, the problem of developing higher-order CE/SE schemes has attracted increasing attention in the literature. Chang developed fourth-order [7] and higher-order [8] CE/SE schemes by using the space–time inversion (STI) invariant property of the original CE/SE scheme and defining new CEs and SEs with which to derive the equations required to calculate the high-order spatial derivatives. Liu et al. [33] developed an arbitrary-order CE/SE scheme based on arbitrary Taylor expansions in the solution elements. In the proposed scheme, the mesh variables were calculated by integrating the conservation law in all of the CEs and calculating the derivatives using a central difference scheme. Chang [34] proposed a novel approach for constructing a highly-stable high-order CE/SE scheme in which the even-order derivatives were calculated by integrating the conservation law in the CEs and the odd-order derivatives were treated using a central difference scheme. In a later study, Bilyeu [35] extended the proposed approach to a system of linear and non-linear hyperbolic partial differential equations.

The present study commences by a short summary of the 1D CE/SE scheme (with $\epsilon = 1/2$) for a non-uniform mesh as a guide (Section 2.2). Then the staggered time marching strategy and the definition of CE and SE for hybrid meshes are illustrated (Section 2.3). Then a second-order space–time conservative scheme is constructed on a hybrid mesh. The time marching scheme for mesh variables is derived by imposing the conservation law on the defined CE with the aid of first Taylor expansion in corresponding SEs (Section 2.4). And the spatial derivatives are evaluated using a finite-difference/weighted-average procedure (Section 2.5). After that, the second-order scheme is extended to three- and fourth-order schemes by using second-order and third-order Taylor expansions in SEs. The definition of CE and SE and the time marching strategies of mesh variables and first-order derivatives are consistent with that of the second-order scheme. To construct an explicit scheme, a finite-difference/weighted-average procedure is designed to derive the high-order derivatives before calculating mesh variables and first-order derivatives (Section 3.2). The robustness and accuracy of the proposed schemes are demonstrated by means of a series of standard benchmark tests (Section 4).

2. Construction of a second-order CE/SE scheme on hybrid mesh

2.1. Governing equation

Consider the following scalar conservation law:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{h} = 0, \tag{1}$$