Journal of Computational Physics 281 (2015) 518–555

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Accuracy-preserving boundary flux quadrature for finite-volume discretization on unstructured grids

Hiroaki Nishikawa *

National Institute of Aerospace, 100 Exploration Way, Hampton, VA 23666-6147, USA

ARTICLE INFO

Article history:
Received 3 April 2014
Received in revised form 16 September 2014
Accepted 15 October 2014
Available online 22 October 2014

Keywords:
Finite-volume
Boundary flux
Boundary condition
High-order
Edge-based discretization
Node-centered
Cell-vertex
Curved boundary

ABSTRACT

In this paper, we derive a general accuracy-preserving boundary flux quadrature formula for the second- and third-order node-centered edge-based finite-volume discretizations on triangular and tetrahedral grids. It is demonstrated that the general formula reduces to some well-known formulas for the second-order scheme in the case of linear fluxes. Some special boundary grids are also examined. In particular, a simple one-point quadrature formula, which is typically used for quadrilateral grids, is shown to be exact for quadratic fluxes on triangular grids with uniformly spaced straight boundaries. Numerical results are presented to demonstrate the accuracy of the general formula, and accuracy deterioration caused by incompatible boundary flux quadrature formulas. In general, the third-order accuracy is lost everywhere in the domain unless the third-order accuracy is maintained at boundary nodes. It is also demonstrated that the third-order scheme does not require high-order curved elements for curved boundaries but requires accurate surface normal vectors defined, which can be estimated by a quadratic interpolation from a given grid, to deliver the designed third-order accuracy.

1. Introduction

Second-order node-centered edge-based finite-volume schemes are widely used in state-of-the-art Computational Fluid Dynamics (CFD) codes, and have been well studied by many researchers in the past decades. The basic algorithm was established in the early work represented by Refs. [1–3], and further studies have been performed in recent work by Refs. [4–8]. One of the key elements in the node-centered formulation is the boundary flux quadrature formula. In the node-centered edge-based discretization, the residual needs to be closed at a boundary node, where a boundary condition is imposed weakly, by integrating the flux along the dual boundary faces. It is well known that the second-order accuracy degrades unless the boundary residual is exact for linear fluxes, i.e., fluxes varying linearly in space. Boundary flux quadrature formulas that ensure the exactness for linear fluxes (linearity-preserving formulas) on triangular and tetrahedral grids have been known for a long time, but a formal proof did not seem available in the literature. Recently, a simple proof was presented in Appendix B in Ref. [9], where the linearity-preserving boundary flux quadrature formulas are derived for triangles and tetrahedra, and extended to quadrilaterals, hexahedra, prisms, and pyramids under certain geometrical regularities.

A recent work by Katz and Sankaran [5,6] revealed that the node-centered edge-based finite-volume discretization achieves third-order accuracy on triangles (tetrahedra in three dimensions) if the nodal gradients are estimated such that they are exact for quadratic functions and both the solution and the flux are linearly extrapolated to the dual-volume

* Tel.: +1 757 864 7244.
E-mail address: hiro@nianet.org.

http://dx.doi.org/10.1016/j.jcp.2014.10.033

face. Since then, there have been efforts for constructing a third-order scheme for the Navier–Stokes equations based on the third-order edge-based scheme [10,11]. However, our experience shows that third-order accuracy is lost unless the boundary residual is designed to be exact for quadratic fluxes. In an attempt to achieve third-order accuracy at boundary nodes, we found that a straightforward application of the linearly-preserving boundary flux quadrature formulas does not provide third-order accuracy and extensions of the proof described in Ref. [9] to third-order accuracy is not straightforward. In this paper, therefore, we propose another approach to derive boundary flux quadrature that preserves the accuracy of the edge-based finite-volume discretization at boundary nodes. The basic idea is to generate a boundary stencil from an interior stencil by shrinking an edge. The procedure has been employed routinely in the residual-distribution method for implementing a weak boundary condition (see, e.g., Ref. [12]), but does not seem to have been utilized in the node-centered edge-based finite-volume method. This paper demonstrates that the approach can be applied to the finite-volume method on triangular and tetrahedral grids, and generates a general formula that preserves the order of accuracy of the interior scheme through the boundary nodes. It is demonstrated that the well-known formula for second-order schemes can be easily derived from the general formula. Extension to tetrahedra is also quite straightforward, and a general formula for tetrahedral grids is also presented in this paper. Numerical results demonstrate that the accuracy-preserving boundary quadrature is critical to the construction of the third-order scheme. Third-order accuracy can be lost completely through the domain by an incompatible boundary flux quadrature formula. Also, it is shown that the third-order finite-volume scheme considered in this paper does not require curved elements for problems involving curved boundaries.

The paper is organized as follows. First, we define the node-centered edge-based finite-volume discretization to be investigated. Second, the accuracy of the discretization is discussed for interior stencils, and the need for boundary fluxes is briefly discussed. Then, we derive a general formula for the boundary flux quadrature that preserves the order of accuracy of the interior discretization at boundary nodes. Some special cases are discussed subsequently. Numerical results are then presented to verify the theoretical results and to examine the impact of low-order formulas. Finally, the paper concludes with remarks.

2. Node-centered edge-based finite-volume discretization

2.1. Discretization

Consider a steady conservation law in two dimensions:

$$\mathrm{div}\,\mathbf{f} = 0, \tag{1}$$

where $\mathbf{f} = (f, g)$ is the flux vector, which is generally a nonlinear function of the solution u. All discussions that follow are based on the scalar equation, but will be valid for systems. Note that the steady conservation law represents various types of equations, including diffusion equations, source terms, time-dependent equations, as they all can be cast in the form of steady conservation laws [13–16]. We consider the node-centered edge-based finite-volume scheme for Eq. (1) in the form:

$$\frac{1}{V_j} \sum_{k \in \{k_j\}} \phi_{jk}(\mathbf{n}_{jk}) = 0, \tag{2}$$

where V_j is the measure of the dual control volume around node j, $\{k_j\}$ is a set of neighbors of j, $\phi_{jk}(\mathbf{n}_{jk})$ is a numerical flux evaluated at the midpoint of the edge $[j,k]$, and $\mathbf{n}_{jk}$ is the scaled directed area vector associated with the edge, i.e., $\mathbf{n}_{jk} = (n_x, n_y) = \mathbf{n}^{\ell}_{jk} + \mathbf{n}^{r}_{jk}$ (see Fig. 1). For the numerical flux, we consider the upwind flux of the form:

$$\phi_{jk}(\mathbf{n}_{jk}) = \frac{1}{2}(\mathbf{f}_L + \mathbf{f}_R) \cdot \hat{\mathbf{n}}_{jk}|\mathbf{n}_{jk}| - \frac{1}{2}|a_n|(u_R - u_L)|\mathbf{n}_{jk}|, \tag{3}$$

where $\hat{\mathbf{n}}_{jk} = (\hat{n}_x, \hat{n}_y)$ denotes the unit directed area vector, and $a_n = (\partial f/\partial u)\hat{n}_x + (\partial g/\partial u)\hat{n}_y$. The left and right solutions and fluxes, $(u_L, u_R, \mathbf{f}_L, \mathbf{f}_R)$, are computed by the linear extrapolation:

$$u_L = u_j + \frac{1}{2}\overline{\nabla} u_j \cdot \Delta\mathbf{r}_{jk}, \qquad u_R = u_k - \frac{1}{2}\overline{\nabla} u_k \cdot \Delta\mathbf{r}_{jk}, \tag{4}$$

$$\mathbf{f}_L = \mathbf{f}_j + \frac{1}{2}\overline{\nabla}\mathbf{f}_j \cdot \Delta\mathbf{r}_{jk}, \qquad \mathbf{f}_R = \mathbf{f}_k - \frac{1}{2}\overline{\nabla}\mathbf{f}_k \cdot \Delta\mathbf{r}_{jk}, \tag{5}$$

where $\Delta\mathbf{r}_{jk} = (\Delta x_{jk}, \Delta y_{jk}) = (x_k - x_j, y_k - y_j)$, and the over-bar indicates that the gradients are approximate, typically provided by a least-squares (LSQ) fit. The flux extrapolation is required for third-order accuracy; it is not necessary for second-order accuracy [6]. In this paper, we consider two LSQ gradient constructions: one with a linear fit, and the other with a quadratic fit. Linear LSQ gradients are exact for linear functions, and quadratic LSQ gradients are exact for quadratic functions. It is known that the edge-based finite-volume discretization yields second-order accuracy with linear LSQ gradients, and third-order accuracy with quadratic LSQ gradients on triangular/tetrahedral grids [6]. The edge-based finite-volume discretization is applicable to and widely used for triangular, quadrilateral, and mixed grids, but here we focus on triangular and tetrahedral grids, only on which third-order accuracy can be achieved.