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Approximation on non-tensor domains including squircles, Part III: Polynomial hyperinterpolation and radial basis function interpolation on Chebyshev-like grids and truncated uniform grids

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ABSTRACT

Single-domain spectral methods have been largely restricted to tensor product bases on a tensor product grid. To break the "tensor barrier", we study approximation in a domain bounded by a "squircle", the zero isoline of $B(x, y) = x^{2\nu} + y^{2\nu} - 1$. The boundary varies smoothly from a circle $[\nu = 1]$ to the square $[\nu = \infty]$. Polynomial least-squares hyperinterpolation converges geometrically as long as the number of points P is (at least) double the number of basis functions N. The polynomial grid was made denser near the boundaries ("Chebyshev-like") by depositing grid points along wisely chosen contours of B. Gaussian radial basis functions (RBFs) were more robust in the sense that they, too, converged geometrically, but hyperinterpolation ($P > N$) and a Chebyshevized grid were unnecessary. A uniform grid, truncated to include only those points within the squircle, was satisfactory even without interpolation points on the boundary (although boundary points are a cost-effective improvement). For a given number of points P, however, RBF interpolation was only slightly more accurate than polynomial hyperinterpolation, and needed twice as many basis functions. Interpolation costs can be greatly reduced by exploiting the invariance of the squircle-bounded domain to the eight elements D_4 dihedral group.

1. Introduction

Spectral methods have been hitherto limited to tensor product domains such as rectangles, disks and triangles. A major research frontier is to extend single domain spectral methods to more complicated geometries. Because this is a very difficult problem, we begin with two simple, simply connected geometries: the domain bounded by a squircle and a quadrifolium perturbed by a disk. (Both are defined and illustrated in Section 2.)

In Part II, we compared two polynomial basis sets. The Chebyshev tensor product is optimum for the square; how good is it for a disk? The Zernike polynomials are optimum for the disk; will they fail for the square? Our conclusion was that, rather to our surprise, there was little difference between the two, although the Chebyshev is less sensitive to the ratio

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Table 1
Notation.

$B(x, y)$	Boundary function: its zero isoline is the boundary
D	Total degree of a polynomial (for x^{m+n}, $D = m + n$)
h	average grid spacing
M	Numbers of points in x (or y) for a tensor product grid on the square
N	Total number of basis functions
P	Total number of interpolation points
q	Parameterization for the squircle: $r = q(\theta; \nu) \equiv \{\cos^{2\nu}(\theta) + \sin^{2\nu}(\theta)\}^{-1/(2\nu)}$
r	Radial coordinate in polar coordinates
R	Radial coordinate in the transformed polar coordinates
T	Computational polar coordinate, the angle
$T_n(x)$	Chebyshev polynomial of degree n
$Z_n^m(x, y)$	Zernike polynomial of angular wavenumber m and total degree n
$Z_n^m(r, \theta)$	Zernike disk polynomial expressed in polar coordinates
α	RBF “shape parameter”; the inverse width relative to the grid spacing h
γ	P/N
$B_j(x)$	Generic basis functions
δ	shape parameter for perturbed quadrifolium boundary curve
$\phi(r)$	RBF Kernel; $\phi(r) \equiv \exp(-[\alpha/h]^2] r^2)$ here
α	relative shape parameter, ϵ divided by average grid spacing
ϵ	absolute shape parameter (inverse width)
θ	Angular coordinate in polar coordinates

of points P to basis functions N. A total degree truncation was used for both where the “total degree” is the sum of the degrees in x and y for reasons explained in Part I [24]. Since the Zernike polynomials are more complicated and much less familiar, all polynomial calculations here will use tensor products of Chebyshev polynomials, $T_m(x)T_n(y)$.

However, radial basis functions have become the “go-to” tool for irregular grid interpolation. A major theme here is to compare polynomial and RBF algorithms: Do RBFs deserve their high reputation?

We will compare the following grids:

1. Truncated uniform square grid without boundary points.
2. Same as the previous but including boundary points.
3. Grids with Chebyshev-like boundary-biasing.

Besides interpolation, we shall also employ “hyperinterpolation” or more precisely, “least-squares hyperinterpolation”. Ian Sloan coined “hyperinterpolation” to describe approximations where the coefficients of the basis functions are computed by numerical quadratures using a set of points whose cardinality exceeds that of the basis set [34]. Here, we do not use quadratures, but rather least-squares approximation with more interpolation conditions than basis coefficients.

If the number of points P equals N, the number of basis functions, the result is “interpolation”. If $P < N$ so that the matrix system is underdetermined, the approximation is “hypointerpolation”. If $P > N$, the result is “hyperinterpolation”.

There are useful strategies that discussed in Part II, but not applied here either because of incompatibility with the D_4 group symmetry (hexagonal grid) [17,2] or to avoid prolonging this article to excessive length (Fourier Extension) [1,21,7,8], RBF Extension [10], mapping the domain to a square or disk through an analytic change of coordinates [28,26]. The Gram–Schmidt process can be applied to construct polynomials orthogonal on the squircle, similar to [27,13] and our experiments will be reported in due course.

Thus, our analysis will be incomplete. Nevertheless, we shall present a variety of new ideas that do break the “rectangularity barrier”.

In Part I [24], we explained how to exploit the D_4 symmetry group. A symmetry-respecting grid can be generated by first creating a grid in a triangular sector bounded by the x-axis and the diagonal line $y = x$. By applying the group reflections, one can populate the rest of the grid. We further showed how to decompose an arbitrary function into six parts which provide the input to six smaller interpolation problems. Lastly, we created linear combinations of the basis functions that are eigenfunctions of the reflections of the symmetry group; we provided the appropriate formulas for radial basis functions, tensor products of Chebyshev polynomials and Zernike polynomials. We will continue to exploit symmetry here, but it will not be a major theme. (Notation is catalogued in Table 1.)

2. Squircles and the perturbed quadrifolium: two simply-connected domains bounded by plane algebraic curves

Definition 1 (*Squircle*). A squircle is the plane algebraic curve defined implicitly as the set of all points satisfying the equation

$$x^{2\nu} + y^{2\nu} = 1 \tag{1}$$

As ν increases, the curve becomes more and more like a square with slightly rounded corners; the limit as $\nu \to \infty$ *is* a square. The word “squircle” is a portmanteau of the words, “square” and “circle”. The squircle is also known as a “Lamé curve” or “Lamé oval”, and also as a “Fermat curve”. It is a special case of the superellipse.