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Compact exponential scheme for the time fractional convection–diffusion reaction equation with variable coefficients

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ABSTRACT

High-order compact exponential finite difference scheme for solving the time fractional convection–diffusion reaction equation with variable coefficients is considered in this paper. The convection, diffusion and reaction coefficients can depend on both the spatial and temporal variables. We begin with the one dimensional problem, and after transforming the original equation to one with diffusion coefficient unity, the new equation is discretized by a compact exponential finite difference scheme, with a high-order approximation for the Caputo time derivative. We prove the solvability of this fully discrete implicit scheme, and analyze its local truncation error. For the fractional equation with constant coefficients, we use Fourier method to prove the stability and utilize matrix analysis as a tool for the error estimate. Then we discuss the two dimensional problem, give the compact ADI scheme with the restriction that besides the time variable, the convection coefficients can only depend on the corresponding spatial variables, respectively. Numerical results are provided to verify the accuracy and efficiency of the proposed algorithm.

1. Introduction

Applications of fractional differential equations (FDEs) have been found in physical, biological, geological and financial systems, and in the recent years there are intensive studies on them. The review article [1] and monograph [2] gave the detailed discussions on fractional differential equations, and a survey of fractional calculus methods in the hydrology can be found [3].

As analytic solutions of the FDEs are available for only very few simple cases, therefore, developing efficient and reliable numerical methods for FDEs is of great interest. Up to now abundant numerical methods have been proposed for solving the space and/or time FDEs, for example, finite difference schemes with convergence of second for the space variable(s) for the time fractional diffusion problems were discussed in [4–7], and the finite difference methods for the fractional differential equations have been reviewed [8]. As the compact finite difference scheme has high-order accuracy and the desirable tridiagonal nature of the finite-difference equations (see [9,10]), it is important to study solving FDEs by this numerical method. For one-dimensional fractional sub-diffusion equation, a compact finite difference scheme with convergence order $\mathcal{O}(\tau)+\mathcal{O}(h^4)$ was recently given [11], a higher order $\mathcal{O}(\tau^{2-\gamma})+\mathcal{O}(h^4)$ one [12], and schemes for variable order FDEs [13]. For two-dimensional problem, the discussions on the compact schemes can be found [14–17]. Finite difference scheme with

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a new fractional numerical differentiation formula, thus having high order accuracy in time, was published recently [18]. For the time fractional sub-diffusion equation with the variable diffusion coefficient, a compact difference scheme is proposed with convergence analysis for solving the Dirichlet boundary value problem [19].

For steady and/or unsteady convection–diffusion equations of integer order, there have been many research papers discussing compact schemes on this topic, e.g., [20,21] and the high-order compact exponential scheme is preferred for solving these kinds of equations [22–25]. Compared with the sub-diffusion problems, solving the time fractional convection–diffusion problems are much more difficult, and there are not many papers on them. Liu et al. gave the complete solution of the time fractional advection–dispersion equation using variable transformation [26], and described an implicit Euler approximation for the time variable fractional order mobile–immobile advection–dispersion model [27]. Saadatmandi et al. [28] presented a collocation method for fractional convection–diffusion equations with variable coefficients. Recently, for the multidimensional time-fractional convection–diffusion equations with constant coefficients and the fractional order lies between 1 and 2, some high order compact schemes are given in [29,30]. In this paper, we use the compact exponential difference scheme to solve the time fractional convection–diffusion reaction problem, and we find that this algorithm is very effective indeed.

The innovations of this paper are in two aspects. First, we consider the problem with variable coefficients for both convection and diffusion terms, so it is a generalization of convection–diffusion problem with constant coefficients in paper [31], and it can be used to solve the fractional Fokker–Planck equation [32,33] whose coefficients are variable. As the main difficulty lies in the variable diffusion coefficient, we use the technique of transforming the original equation to a new one with constant diffusion coefficient. The second is that though we can only give the convergence analysis for the case of constant coefficients, we include the reaction term and give the convergence result in the discrete l^2 norm here using the matrix analysis.

The model problem considered is the time fractional convection–diffusion reaction equation with variable coefficients,

$$ {}_0^C D_t^\gamma u(x,t) - \big(a(x,t)u_x\big)_x + b(x,t)u_x + c(x,t)u(x,t) = f(x,t), \quad L_1 < x < L_2,\ 0 < t < T. \tag{1} $$

Here the coefficients a, b and c are functions depending on x and t, $c(x,t) \ge 0$ and $a(x,t) \ge a_0 > 0$ with a_0 being a constant. The Caputo fractional derivative ${}_0^C D_t^\gamma v$ $(0 < \gamma < 1)$ of the function $v(x,t)$ is defined by [2], i.e.,

$$ {}_0^C D_t^\gamma v = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial v(x,\tau)}{\partial \tau} \cdot \frac{1}{(t-\tau)^\gamma} d\tau. $$

The initial condition for (1) is

$$ u(x,0) = w(x), \quad L_1 < x < L_2, \tag{2} $$

with the Dirichlet boundary conditions given by

$$ u(L_1,t) = \varphi_1(t), \qquad u(L_2,t) = \varphi_2(t), \quad t \ge 0, \tag{3} $$

and we assume that the true solution of (1)–(3) has sufficient smoothness for discretization and error estimate.

The paper is organized as follows. In Section 2, as the variable diffusion coefficient causes difficulty in solving this kind of equations, so we use the skill of transforming the original equation to one with diffusion coefficient unity. We adopt the high-order exponential (HOE) scheme for the steady problems first, then we use the high order discretization for the time fractional derivative to give an implicit compact exponential difference scheme for the fractional convection–diffusion-reaction equation. The local truncation error is discussed, and for the equation with constant coefficients, the unconditionally stability for the initial values is proved by Fourier method, with the error estimate obtained by the matrix analysis in Section 3. In Section 4, we discuss the two dimensional problem, give the compact ADI scheme to reduce the computational cost. We assume that the advection term takes the form that the first order spatial derivatives are multiplied by the coefficients depending on the time variable and the corresponding spatial variables. Finally, some numerical examples are given in Section 5 to verify the theoretical conclusions. This paper closes with a summary in Section 6.

2. High-order compact exponential difference scheme

2.1. Partition of the domain and some one-dimensional vectors

For the numerical solution of (1)–(3) we introduce a uniform grid of mesh points (x_i, t_n), with $x_i = L_1 + ih$, $i = 0, 1, \cdots, N_x + 1$, and $t_n = n\tau$, $n = 0, 1, \cdots, N$. Here N_x and N are positive integers, $h = (L_2 - L_1)/(N_x + 1)$ is the mesh-width in x, and $\tau = T/N$ is the time step. For any function $v(x,t)$, we let $v_i^n = v(x_i, t_n)$, e.g., the theoretical solution u at the mesh point (x_i, t_n) is denoted by u_i^n, and U_i^n stands for the solution of an approximating difference scheme at the same mesh point. On each time level t_n the exact solution vector of order N_x is denoted by $\mathbf{u}^n = \mathbf{u}(t_n) = (u_1^n, u_2^n, \cdots, u_{N_x}^n)^T$, the approximate solution vector $\mathbf{U}^n = \mathbf{U}(t_n) = (U_1^n, U_2^n, \cdots, U_{N_x}^n)^T$, and we put $\mathbf{F}^n = \mathbf{F}(t_n) = (f_1^n, f_2^n, \cdots, f_{N_x}^n)^T$.