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A new difference scheme for the time fractional diffusion equation

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ABSTRACT

In this paper we construct a new difference analog of the Caputo fractional derivative (called the $L2\text{-}1_\sigma$ formula). The basic properties of this difference operator are investigated and on its basis some difference schemes generating approximations of the second and fourth order in space and the second order in time for the time fractional diffusion equation with variable coefficients are considered. Stability of the suggested schemes and also their convergence in the grid L_2-norm with the rate equal to the order of the approximation error are proved. The obtained results are supported by the numerical calculations carried out for some test problems.

1. Introduction

Recently a noticeable growth of the attention of researches to the fractional differential equations has been observed. It is caused by numerous effective applications of fractional calculation to various areas of science and engineering [1–6]. For example, mathematical language of fractional derivatives is irreplaceable for the description of the physical process of statistical transfer and, as it is known, leads to diffusion equations of fractional orders [7,8].

Consider the time fractional diffusion equation with variable coefficients

$$\partial_{0t}^{\alpha} u(x,t) = \mathcal{L}u(x,t) + f(x,t), \quad 0 < x < l,\ 0 < t \le T, \tag{1}$$

$$u(0,t) = 0, \qquad u(l,t) = 0, \quad 0 \le t \le T, \qquad u(x,0) = u_0(x), \quad 0 \le x \le l, \tag{2}$$

where

$$\partial_{0t}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\eta)}{\partial \eta} (t-\eta)^{-\alpha} d\eta, \quad 0 < \alpha < 1 \tag{3}$$

is the Caputo derivative of the order α,

$$\mathcal{L}u(x,t) = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) - q(x,t)u,$$

$k(x,t) \ge c_1 > 0$, $q(x,t) \ge 0$ and $f(x,t)$ are sufficiently smooth functions.

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The time fractional diffusion equation represents a linear integro-differential equation. Its solution not always can be found analytically; therefore it is necessary to use numerical methods. However, unlike the classical case, we require information about all the previous time layers, when numerically approximating a time fractional diffusion equation on a certain time layer. For that reason algorithms for solving the time fractional diffusion equations are rather time-consuming even in one-dimensional case. Upon transition to two-dimensional and three-dimensional problems their complexity considerably increases. In this regard constructing stable differential schemes of higher order approximation is a very important task.

A widespread difference approximation of fractional derivative (3) is the so-called $L1$ method [2,9] which is defined as follows

$$\partial_{0t_{j+1}}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{j} \frac{u(x,t_{s+1}) - u(x,t_s)}{t_{s+1}-t_s} \int_{t_s}^{t_{s+1}} \frac{d\eta}{(t_{j+1}-\eta)^{\alpha}} + r^{j+1}, \tag{4}$$

where $0 = t_0 < t_1 < \ldots < t_{j+1}$, and r^{j+1} is the local truncation error. In the case of the uniform mesh, $\tau = t_{s+1} - t_s$, for all $s = 0, 1, \ldots, j+1$, it was proved that $r^{j+1} = \mathcal{O}(\tau^{2-\alpha})$ [10–12]. The $L1$ method has been widely used for solving the fractional differential equations with Caputo derivatives [10–16].

Difference schemes of the increased order of approximation such as the compact difference scheme [14,17–19] and spectral method [11,20,21] were applied to improve the spatial accuracy of fractional diffusion equations. However, it is rather difficult to get a high-order time approximation due to the singularity of fractional derivatives.

A good approximation of the $L1$ method is observed in case of a nonuniform mesh, when it is refined in a neighborhood of the point t_{j+1} [9]. Though the nonuniform mesh turns out to be more effective in comparison with the uniform one, it will not generate the second order of approximation in all points of the mesh.

In [22] a new difference analog of the Caputo fractional derivative with the order of approximation $\mathcal{O}(\tau^{3-\alpha})$, called $L1$-2 formula, is constructed. On the basis of this formula calculations of difference schemes for the time-fractional sub-diffusion equations in bounded and unbounded spatial domains and the fractional ODEs are carried out. If the stability and convergence of difference schemes from [22] will be strictly proved, then this will undoubtedly be a significant progress in computing the time-fractional partial differential equations.

Using the energy inequality method, a priori estimates for the solution of the Dirichlet and Robin boundary value problems for the diffusion-wave equation with Caputo fractional derivative have been obtained in [15,23].

In this paper a new difference analog of the fractional Caputo derivative with the order of approximation $\mathcal{O}(\tau^{3-\alpha})$ for each $\alpha \in (0,1)$ is constructed. Properties of the obtained difference operator are studied. Difference schemes of the second and fourth order of approximation in space and the second order in time for the time fractional diffusion equation with variable coefficients are constructed. Using the method of energy inequalities, the stability and convergence of these schemes in the mesh L_2-norm are proved. Numerical calculations of some test problems confirming reliability of the obtained results are carried out.

2. Family of difference schemes. Stability and convergence

In this section, families of difference schemes in a general form set on a non-uniform time mesh are investigated. A criterion of the stability of the difference schemes in the mesh L_2-norm is obtained. The convergence of solutions of the difference schemes to the solution of the corresponding differential problem with the rate equal to the order of the approximation error is proved.

2.1. Family of difference schemes

In the rectangle $\overline{Q}_T = \{(x,t) : 0 \le x \le l,\ \ 0 \le t \le T\}$ we introduce the mesh $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$, where $\overline{\omega}_h = \{x_i = ih,\ i = 0, 1, \ldots, N;\ hN = l\}$, $\overline{\omega}_\tau = \{t_j :\ 0 = t_0 < t_1 < t_2 < \ldots < t_{M-1} < t_M = T\}$.

Basically the family of difference schemes, approximating problem (1)–(2) on the mesh $\overline{\omega}_{h\tau}$, has the form

$$ {}_g\Delta_{0t_{j+1}}^{\alpha} y_i = \Lambda y_i^{(\sigma_{j+1})} + \varphi_i^{j+1}, \quad i = 1, 2, \ldots, N-1,\ j = 0, 1, \ldots, M-1, \tag{5}$$

$$ y(0,t) = 0, \qquad y(l,t) = 0, \quad t \in \overline{\omega}_\tau, \qquad y(x,0) = u_0(x), \quad x \in \overline{\omega}_h, \tag{6}$$

where

$$ {}_g\Delta_{0t_{j+1}}^{\alpha} y_i = \sum_{s=0}^{j} \left(y_i^{s+1} - y_i^s\right) g_s^{j+1}, \quad g_s^{j+1} > 0, \tag{7}$$

is a difference analog of the Caputo derivative of the order α $(0 < \alpha < 1)$, Λ is a difference operator approximating the continuous operator $\mathcal{L}$, such that the operator $-\Lambda$ preserves its positive definiteness ($(-\Lambda y, y) \ge \varkappa \|y\|^2$, $\varkappa > 0$), for example

$$ (\Lambda y)_i = \left((a y_{\bar{x}})_x - d y\right)_i = \frac{a_{i+1} y_{i+1} - (a_{i+1} + a_i) y_i + a_i y_{i-1}}{h^2} - d_i y_i, \tag{8}$$