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Stability and convergence of finite difference schemes for a class of time-fractional sub-diffusion equations based on certain superconvergence ☆

Guang-Hua Gao [a], Hai-Wei Sun [b,*], Zhi-Zhong Sun [c]

[a] College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210046, PR China
[b] Department of Mathematics, University of Macau, Macao
[c] Department of Mathematics, Southeast University, Nanjing 210096, PR China

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ABSTRACT

This paper is devoted to the construction and analysis of finite difference methods for solving a class of time-fractional subdiffusion equations. Based on the certain superconvergence at some particular points of the fractional derivative by the traditional first-order Grünwald–Letnikov formula, some effective finite difference schemes are derived. The obtained schemes can achieve the global second-order numerical accuracy in time, which is independent of the values of anomalous diffusion exponent α $(0 < \alpha < 1)$ in the governing equation. The spatial second-order scheme and the spatial fourth-order compact scheme, respectively, are established for the one-dimensional problem along with the strict analysis on the unconditional stability and convergence of these schemes by the discrete energy method. Furthermore, the extension to the two-dimensional case is also considered. Numerical experiments support the correctness of the theoretical analysis and effectiveness of the new developed difference schemes.

1. Introduction

In the recent few decades, the remarkable applications of fractional calculus in diverse engineering fields have been gradually realized and meanwhile, the discussion on the related fractional differential equations becomes a hot topic of many scholars. To seek the exact solutions to these differential equations is not an easy job in spite of some research results around the world on the subject [1–4]. Effective and simple numerical methods for solving these equations tend to be favored in practical computations, for instance, readers can refer to the works [5–14].

Anomalous diffusion equations are often used to describe the transport dynamics in various complex systems where Gaussian statistics are no longer followed and the Fick second law fails to describe the related transport behaviors. Anomalous diffusion in the presence of an external velocity or force field has been modeled in numerous ways, one of which is given in terms of continuous time random walk (CTRW) models. Based on the CTRW models, a generalized diffusion

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* Corresponding author.
E-mail addresses: gaoguanghua1107@163.com (G.-H. Gao), HSun@umac.mo (H.-W. Sun), zzsun@seu.edu.cn (Z.-Z. Sun).

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equation of fractional order is derived in [15]. Until now, two forms of this kind of equations often have appeared: one is written as [16–18]

$$ {}_0^C\mathcal{D}_t^{\alpha} u(x,t) = u_{xx}(x,t) + f(x,t), \quad (x,t) \in (a,b) \times (0,T], \tag{1.1}$$

where $0 < \alpha < 1$ and ${}_0^C\mathcal{D}_t^{\alpha}$ is the α-th order Caputo time-fractional operator defined by

$$ {}_0^C\mathcal{D}_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(x,s)\,\mathrm{d}s, $$

and we call it the time Caputo-type subdiffusion case; the other takes the form of [19–21,23,26]

$$ u_t(x,t) = {}_0^{RL}\mathcal{D}_t^{1-\alpha} u_{xx}(x,t) + g(x,t), \quad (x,t) \in (a,b) \times (0,T], \tag{1.2}$$

where $0 < \alpha < 1$ and ${}_0^{RL}\mathcal{D}_t^{\alpha}$ is the α-th order Riemann–Liouville fractional operator defined by

$$ {}_0^{RL}\mathcal{D}_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(x,s)\,\mathrm{d}s, $$

and we call it the time Riemann–Liouville-type subdiffusion equation. These two forms are equivalent under some regularity assumption for $u(x,t)$ in time and $g(x,t) = {}_0^{RL}\mathcal{D}_t^{1-\alpha} f(x,t)$; see, for details, [27].

Traditionally, the direct difference approximation for the time-fractional derivative covers two ways: the $L1$ formula and the Grünwald–Letnikov formula. The former is based on the piecewise linear interpolating approximation with respect to t for the integrand $u(x,t)$ inside the integral in the Caputo fractional derivative sense, while the latter is often used to handle the Riemann–Liouville time-fractional derivative. For the α-th ($0 < \alpha < 1$) fractional derivative, the numerical accuracy of $L1$ formula is proved to be $2 - \alpha$ [16,17], which is less than two, and that of the Grünwald–Letnikov formula depends on the choice of generating function. The common generating function with coefficients in this formula is chosen to be $(1-z)^{\alpha}$, and only the first-order accuracy is attained [20–25].

Recently, the great efforts to enhance the numerical accuracy of approximating time-fractional derivatives have been made. Cao and Xu [28] started from the equivalent Volterra integral form of the original time Caputo-type fractional differential equations to design the high order numerical scheme in time. The computation of coefficients in related numerical methods is quite complex and expensive. Gao, Sun and Zhang [29] presented a modified $L1$ numerical differentiation formula to directly discretize the Caputo time-fractional derivative and higher-order numerical accuracy seems to be realized, where the strict convergence analysis for the corresponding difference scheme has not been available. Wang and Vong [30] established some difference schemes with the second-order accuracy in time for solving the time-fractional subdiffusion and diffusion-wave equations by mean of weighed Grünwald–Letnikov formula. The similar techniques were used to deal with the problem with Neumann boundary conditions in [31]. Ding and Li [32] proposed a second-order difference approximation for the Riemann–Liouville time-fractional derivative by the Grünwald–Letnikov formula with the generating function $(3/2 - 2z + z^2/2)^{\alpha}$. The obtained coefficients were more complex than the usual first-order Grünwald–Letnikov formula. Here, we shall show the alternative way to design the high-order finite difference scheme using the simple first-order Grünwald–Letnikov formula to approximate the Riemann–Liouville time-fractional derivative. Zhao and Deng [34] investigated a series of high order pseudo-compact schemes for space fractional diffusion equations based on the superconvergent approximations for fractional derivatives.

The key point of this approach is based on the significant work by Nasir et al. [33], where the superconvergent points of the first-order Grünwald–Letnikov formula to approximate the Riemann–Liouville time-fractional derivative are exactly pinpointed. Namely, the standard first-order Grünwald–Letnikov formula or shifted first-order Grünwald–Letnikov formula to approximate the fractional derivative value at current point is only first-order accurate, whereas one-order higher numerical accuracy at some shifted positions can be obtained. To our knowledge, the technique using such superconvergence to directly improve the numerical accuracy of approximating the time-fractional derivative has not appeared in the literature, except the very recent work by Dimitrov in [35] for the one-dimensional time Caputo-type subdiffusion case. In order to clarify the success of this idea, we begin with the one-dimensional problem and construct a second-order accurate difference scheme both in time and space for solving the time-fractional sub-diffusion equation. Then the spatial fourth-order compact scheme is also established along the similar route. Furthermore, the extension to the two-dimensional case is also taken into account.

The outline of this paper is as follows. In Section 2, a second-order difference scheme both in time and space is derived by considering the governing equation at the superconvergent point of the standard first-order Grünwald–Letnikov formula for the Riemann–Liouville derivative. The unconditional stability and convergence of the second-order scheme are proved in Section 3 by the discrete energy method. In Section 4, the compact difference scheme with the convergence of second order in time and fourth order in space is constructed. The stability and convergence are also given. Section 5 is devoted to the discussion of the two-dimensional case. Numerical examples are included in Section 6 to verify the efficiency of the proposed schemes. A brief conclusion ends this work finally.