Journal of Computational Physics 280 (2015) 563–569

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Short note

Anomaly of the lattice Boltzmann methods in three-dimensional cylindrical flows

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A R T I C L E   I N F O

Article history:
Received 13 February 2014
Received in revised form 27 August 2014
Accepted 3 October 2014
Available online 8 October 2014

Keywords:
Lattice Boltzmann equation
Rotational invariance
Turbulent flow
Error analysis

A B S T R A C T

Truncation error terms of the lattice Boltzmann equation (LBE) are analysed for three-dimensional flow simulations by the D3Q15, D3Q19 and D3Q27 models. By applying the analysis of Holdych et al. (2004) [9], rotational invariance of the Reynolds averaged error terms is examined in cylindrical coordinates. It is found that error terms distribute cyclically in the circumferential direction and produce false forces in particular directions by the D3Q15 and D3Q19 models. When flow becomes turbulent, this tendency is reinforced by the Reynolds stress resulting in the generation of obvious spurious currents. By the D3Q27 model, it is confirmed that the LBE keeps rotational invariance in laminar flows whilst non-axisymmetric error terms arise in turbulent flows. However, the magnitudes of the error terms are less than two percent of those by the D3Q15 and D3Q19 models and hence unphysical spurious currents are sufficiently weak.

1. Introduction

Recently, the lattice Boltzmann method has become a popular tool for simulating turbulent flows (e.g., [1–4]). However, it has been reported that insufficient isotropy is sometimes visible in simulating turbulent axisymmetric flows (e.g., [5–7]). Indeed, Kang and Hassan [6] reported that the D3Q19 discrete velocity model produced unphysical spurious currents in turbulent pipe flows and violated Galilean invariance in square duct turbulence while the D3Q27 model did not. They also confirmed that the grid resolution was not the main reason for the spurious currents. Even in a laminar flow, similar anomaly was reported by White and Chong [8] who obtained non-axisymmetric flow patterns in a circular nozzle-throat flow at $\mathrm{Re} \leq 500$ by the D3Q15 and D3Q19 models but not by the D3Q27 model. Those authors considered that the flow physics was not properly replicated and the defective planes had a deficiency in the momentum transfer when the lattice planes were not fully isotropic as in the D3Q15 and D3Q19 models. However, the mathematical background and mechanism for those phenomena have not been understood yet. Therefore, to understand the reason for those anomalous behaviours, this study examines rotational invariance of the error terms by applying the analysis of Holdych et al. [9] to cylindrical coordinates.

2. Truncation error of the lattice Boltzmann equation

The lattice Boltzmann equation (LBE) can be obtained by discretizing the velocity space of the Boltzmann equation into a finite number of discrete velocities $\mathbf{w}_\alpha\{\alpha = 0, \cdots, Q-1\}$. There are several discrete velocity models for three-dimensional

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http://dx.doi.org/10.1016/j.jcp.2014.10.002

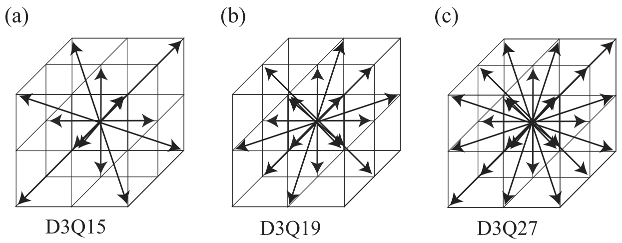

Fig. 1. Discrete models of a square grid lattice Boltzmann method: (a) D3Q15, (b) D3Q19, (c) D3Q27.

(3D) flows such as the D3Q15, D3Q19 and D3Q27 models for the standard square lattices as shown in Fig. 1. The most common evolution equation of the particle distribution function $\boldsymbol{f}_\alpha$ is the lattice BGK model:

$$\boldsymbol{f}_\alpha(\mathbf{x}+\mathbf{w}_\alpha\Delta x, t+\Delta t) = \boldsymbol{f}_\alpha(\mathbf{x}, t) - \frac{1}{\tau}\left[\boldsymbol{f}_\alpha(\mathbf{x}, t) - \boldsymbol{f}_\alpha^{eq}(\mathbf{x}, t)\right], \tag{1}$$

where Δx, Δt, τ and $\boldsymbol{f}_\alpha^{eq}$ are the lattice spacing, the time step, the relaxation time and the equilibrium distribution function, respectively. Holdych et al. [9] modified Eq. (1) to

$$\boldsymbol{f}_\alpha(\mathbf{x}, t) = \left(1-\frac{1}{\tau}\right)\boldsymbol{f}_\alpha(\mathbf{x}-\mathbf{w}_\alpha\Delta x, t-\Delta t) + \frac{1}{\tau}\boldsymbol{f}_\alpha^{eq}(\mathbf{x}-\mathbf{w}_\alpha\Delta x, t-\Delta t), \tag{2}$$

and applied this expression recursively to eliminate $\boldsymbol{f}_\alpha$ from the right hand side of Eq. (2):

$$\boldsymbol{f}_\alpha(\mathbf{x}, t) = \frac{1}{\tau}\sum_{n=1}^{\infty}\left(1-\frac{1}{\tau}\right)^{n-1}\boldsymbol{f}_\alpha^{eq}(\mathbf{x}-n\mathbf{w}_\alpha\Delta x, t-n\Delta t). \tag{3}$$

Eq. (3) was then applied to the relations between the distribution function and macroscopic flow quantities: the pressure p and the velocity $\mathbf{u}$, as

$$p(\mathbf{x}, t) = \frac{c_s^2}{\tau}\sum_{\alpha=0}^{Q-1}\sum_{n=1}^{\infty}\left(1-\frac{1}{\tau}\right)^{n-1}\boldsymbol{f}_\alpha^{eq}(\mathbf{x}-n\mathbf{w}_\alpha\Delta x, t-n\Delta t), \tag{4}$$

$$\rho_0\mathbf{u}(\mathbf{x}, t) = \left(\frac{\Delta x}{\Delta t}\right)\frac{1}{\tau}\sum_{\alpha=0}^{Q-1}\sum_{n=1}^{\infty}\left(1-\frac{1}{\tau}\right)^{n-1}\mathbf{w}_\alpha\boldsymbol{f}_\alpha^{eq}(\mathbf{x}-n\mathbf{w}_\alpha\Delta x, t-n\Delta t), \tag{5}$$

where c_s is the sound speed and the density ρ is expressed as the sum of constant and fluctuation values: $\rho = \rho_0 + \delta\rho$ for the LBE of the incompressible flows [10]. By applying the Taylor series expansion to the equilibrium distribution function as

$$\boldsymbol{f}_\alpha^{eq}(\mathbf{x}-n\mathbf{w}_\alpha\Delta x, t-n\Delta t) = \boldsymbol{f}_\alpha^{eq}(\mathbf{x}, t) + \sum_{m=1}^{\infty}\frac{1}{m!}(-n\Delta t\partial_t - n\Delta x\mathbf{w}_\alpha\cdot\nabla)^m\boldsymbol{f}_\alpha^{eq}(\mathbf{x}, t), \tag{6}$$

Eqs. (4) and (5) can form the incompressible Navier–Stokes equations. With the equilibrium distribution function of He and Luo [10]:

$$\boldsymbol{f}_\alpha^{eq}(\mathbf{x}, t) = \varpi_\alpha\left(\rho + \rho_0\left[\frac{\Delta x}{\Delta t}\frac{\mathbf{w}_\alpha\cdot\mathbf{u}}{c_s^2} + \frac{\Delta x^2}{\Delta t^2}\frac{(\mathbf{w}_\alpha\cdot\mathbf{u})^2}{2c_s^4} - \frac{\mathbf{u}^2}{2c_s^2}\right]\right), \tag{7}$$

the resultant normalized equations are

- the continuity equation:

$$\partial_i u_i = 0 + \mathcal{E}_\Theta, \tag{8}$$

- the momentum equation:

$$\partial_t u_i + u_j\partial_j u_i = -\partial_i p + \frac{1}{\mathrm{Re}}\partial_{jj}u_i + \mathcal{E}_i, \tag{9}$$

where $\mathcal{E}_\Theta$ and $\mathcal{E}_i$ are the truncation errors of the continuity and momentum equations, respectively. (The notations for the partial differentiations are $\partial_i = \partial/\partial x_i$, $\partial_{ij} = \partial^2/(\partial x_i\partial x_j)$, etc.) In cases of the standard square lattices, Holdych et al. [9] derived those truncation errors as

$$\mathcal{E}_\Theta = -\Delta x^2\frac{\mathrm{Re}^2(2\tau-1)^2}{12}\partial_t p + O\left(\Delta x^4\right), \tag{10}$$