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Computation of potentials from current electrodes in cylindrically stratified media: A stable, rescaled semi-analytical formulation

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ABSTRACT

We present an efficient and robust semi-analytical formulation to compute the electric potential due to arbitrary-located point electrodes in three-dimensional cylindrically stratified media, where the radial thickness and the medium resistivity of each cylindrical layer can vary by many orders of magnitude. A basic roadblock for robust potential computations in such scenarios is the poor scaling of modified-Bessel functions used for computation of the semi-analytical solution, for extreme arguments and/or orders. To accommodate this, we construct a set of rescaled versions of modified-Bessel functions, which avoids underflows and overflows in finite precision arithmetic, and minimizes round-off errors. In addition, several extrapolation methods are applied and compared to expedite the numerical evaluation of the (otherwise slowly convergent) associated Sommerfeld-type integrals. The proposed algorithm is verified in a number of scenarios relevant to geophysical exploration, but the general formulation presented is also applicable to other problems governed by Poisson equation such as Newtonian gravity, heat flow, and potential flow in fluid mechanics, involving cylindrically stratified environments.

1. Introduction

Resistivity logging is extensively used for detecting, characterizing, and analyzing hydrocarbon-bearing zones in the subsurface earth [30,17,7,4,13,14,35,3]. This sensing modality employs electrode-type devices mounted on a mandrel that inject electric currents into the surrounding earth formation [5,29]. The ensuing electric potential is then measured at different locations to provide estimates for the surrounding resistivity. Many brute-force numerical techniques such as finite-differences, finite elements, numerical mode-matching, and finite volumes method can be used to model the response of resistivity logging tools [12,8,10,26,23,24,20,25,21,22,11,6,9]. Brute-force techniques are rather versatile and applicable to arbitrary resistivity distributions; however, at the same time, this precludes optimality in particular cases of special interest, such as when resistivity logging environment can be represented as a cylindrically stratified medium [2]. Depending on the implementation, brute-force techniques may have difficulties handling extreme sharp discontinuities in the coefficients, as is the case for the resistivity parameter for the physical scenario considered here, which can change by many orders of magnitude across adjacent layers.

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In this paper, a robust semi-analytical formulation for computing the electric potential due to arbitrary-located point electrodes in three-dimensional cylindrically stratified media is proposed. The present formulation is based on a series expansion in terms of azimuth Fourier modes and a spectral integral over the vertical wavenumber along the axial direction. The resulting problem in terms of the radial variable yields a set of modified Bessel equations. The present formulation removes roadblocks for numerical computations associated with the poor scaling of modified-Bessel functions for very small and/or very large arguments and/or orders [27,31,1]. This is done by constructing a set of rescaled, modified-Bessel functions that can be stably evaluated under double precision arithmetic, akin to what has been done in the past for ordinary (non-modified) Bessel functions [16]. The present formulation also carefully manipulates the analytical formulae for the potential in such media to yield a set of integrand expressions that can be computed in a robust manner under double precision for a wide range of layer thicknesses, layer resistivities, and source and observation point separations. Finally, a number of acceleration techniques are implemented and compared for the efficient numerical integration of the Sommerfeld-type (spectral) integrals, which otherwise suffer from slow convergence. The proposed algorithm is verified in a number of practical scenarios relevant to geophysical exploration. More generally, the mathematical setting here corresponds to the classical problem of obtaining the Green's function for the steady diffusion equation (Poisson problem) with discontinuous coefficients in a separable geometry. As such, the general formalism presented here is also applicable to other problems governed by Poisson equation such as Newtonian gravity, heat flow, elasticity, neutron transport, and potential flows in fluid mechanics, in cylindrically stratified geometries.

2. Formulation

2.1. Electric potential in homogeneous media

In a homogeneous medium, the electric potential ψ from a current electrode at the origin writes as [31]

$$\psi = \frac{\mathcal{I}}{4\pi\sigma\sqrt{\rho^2+z^2}} = \frac{\mathcal{I}}{2\pi^2\sigma}\int_0^\infty K_0(\lambda\rho)\cos(\lambda z)d\lambda, \tag{1}$$

where $\mathcal{I}$ is the electric current flowing into the medium from the electrode, σ is the conductivity of the medium, and $K_0(\cdot)$ is the modified-Bessel function of the second kind of the zeroth order. For the second equality, the complete Lipschitz–Hankel integral [32] is employed. When the source is off the origin, higher order azimuthal modes appear. Using the addition theorem for K_0, (1) is modified to

$$\begin{aligned}\psi &= \frac{\mathcal{I}}{2\pi^2\sigma}\int_0^\infty K_0\big(\lambda|\boldsymbol{\rho}-\boldsymbol{\rho}'|\big)\cos\big(\lambda(z-z')\big)d\lambda \\ &= \frac{\mathcal{I}}{2\pi^2\sigma}\sum_{n=-\infty}^{\infty} e^{\mathrm{i}n(\phi-\phi')}\int_0^\infty I_n(\lambda\rho_<)K_n(\lambda\rho_>)\cos\big(\lambda(z-z')\big)d\lambda,\end{aligned} \tag{2}$$

in terms of modified-Bessel functions of both first, $I_n(\cdot)$, and second, $K_n(\cdot)$, kinds. In the above, primed coordinates (ρ', ϕ', z') represent the source location and unprimed coordinates (ρ, ϕ, z) represent the observation point. Also, $\rho_< = \min(\rho, \rho')$ and $\rho_> = \max(\rho, \rho')$.

2.2. Electric potential in cylindrically stratified media

In a cylindrically stratified medium, boundary conditions at the interfaces need to be incorporated. Let us first consider the case with two distinct cylindrical layers, as depicted in Fig. 1. When the source is embedded in layer 1, we denote it the *outgoing-potential* case. In this case, the primary potential ψ^p is a function of $K_n(\lambda\rho)$ because $I_n(\lambda\rho)$ diverges as ρ goes to infinity. On the other hand, when the source is embedded in layer 2, we denote it the *standing-potential* case and ψ^p is a function of $I_n(\lambda\rho)$ instead of $K_n(\lambda\rho)$ because $K_n(\lambda\rho)$ diverges when ρ goes to zero. For the outgoing-potential case, the n-th harmonic with $e^{\mathrm{i}n(\phi-\phi')}$ dependence in layer 1 and layer 2 can be expressed, resp., as

$$\psi_1 = \big[K_n(\lambda\rho) + R_{12}I_n(\lambda\rho)\big]A_0, \tag{3a}$$

$$\psi_2 = T_{12}K_n(\lambda\rho)A_0, \tag{3b}$$

where R_{12} and T_{12} are the (local) reflection and transmission coefficients at the boundary a_1, and A_0 is an arbitrary amplitude of the primary potential. Applying the boundary conditions [31] at the interface, we obtain

$$R_{12} = \frac{(\sigma_2-\sigma_1)K_n(\lambda a_1)K_n'(\lambda a_1)}{\sigma_1 I_n'(\lambda a_1)K_n(\lambda a_1) - \sigma_2 I_n(\lambda a_1)K_n'(\lambda a_1)}, \tag{4a}$$

$$T_{12} = \frac{\sigma_1}{\lambda a_1[\sigma_1 I_n'(\lambda a_1)K_n(\lambda a_1) - \sigma_2 I_n(\lambda a_1)K_n'(\lambda a_1)]}. \tag{4b}$$