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Modified level set equation and its numerical assessment

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ABSTRACT

In the context of level set methods, the level set equation is modified by embedding a source term. The exact expression of this term is such that the eikonal equation is automatically satisfied, and also, this term is zero on the interface. Theoretically, it renders the reinitialization of level sets unnecessary, similarly to the extension velocity method. The exact expression of the source term makes also possible the derivation of its local approximate forms, of zero-, first- and higher-order accuracy. Application of those forms simplifies the realization of level set methods in comparison with the extension velocity method, but requires the return to the reinitialization procedure. Nevertheless, the advantage of local approximate forms of the proposed source term is that the number of reinitializations can be significantly reduced in comparison with the standard level set equation with the reinitialization procedure. Furthermore, with increasing the order of accuracy of approximation less number of reinitializations is needed. This leads to improvement of the interface resolution. The paper describes the new approach and an assessment of its performance in different test cases.

1. Introduction

In a variety of physical processes the discontinuity in physical properties is mimicked by the evolution of a fluid-interface. Examples include immiscible gas–liquid flows, premixed flames, solidification and melting phenomena, etc. In these examples, the level set methods are often used for simulation of the moving interface. The most basic description of these methods, pioneered by Osher and Sethian in [1], can be found in books [2–4]. Essential is that the interface is embedded as the zero level set $\Sigma = \{\mathbf{x} : G(\mathbf{x}, t) = 0\}$ of a continuous level set function $G(\mathbf{x}, t)$ evolving according to the following field-equation:

$$\frac{\partial G}{\partial t} + \mathbf{u} \cdot \nabla G = 0. \tag{1}$$

This equation, with a given initial distribution $G(\mathbf{x}, t)|_{t=0} = G_0(\mathbf{x})$, is often referred to as the level set equation, or the G-equation. Here $\mathbf{u}(\mathbf{x}, t)$ is the flow velocity field. Geometric quantities such as the unit vector $\mathbf{n}$, normal to the interface, and the interface curvature κ can be determined from the level set field:

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$$\mathbf{n} = \frac{\nabla G}{|\nabla G|}, \qquad \kappa = \nabla \cdot \mathbf{n}. \tag{2}$$

According to definition (2), the equivalent form of Eq. (1) can be written in terms of the normal velocity $F = \mathbf{u} \cdot \mathbf{n}$:

$$\frac{\partial G}{\partial t} + F|\nabla G| = 0. \tag{3}$$

The well-known problem addressed to Eq. (1) is this: if the flow velocity is not constant, the level set function G may become strongly distorted, and then the numerical integration of (1) suffers from loss of accuracy in the prediction of the interface. Consequently, the relevant geometric quantities (2) are poorly computed. In level set methods this problem is remedied by the *reinitialization procedure*[1] [5], i.e. by reconstructing the level set function such that it satisfies the eikonal equation:

$$|\nabla G| = 1, \qquad G|_{\mathbf{x} \in \Sigma} = 0. \tag{4}$$

There are a number of numerical methods for reinitialization (or redistancing) of a given level set function to the corresponding signed distance function. The most popular methods fast marching methods [6–8], fast sweeping methods [9,10] and reinitialization methods [11–18] based on PDEs (partial differential equation). In this work, we address to the PDE-based reinitialization method, referred to as the *reinitialization procedure*. It relies on the evolutional form of Eq. (4) through an iterative process:

$$\frac{\partial G}{\partial \tau} = \operatorname{sgn}(\tilde{G})\big(1 - |\nabla G|\big), \qquad G(\mathbf{x}, \tau = 0) = \tilde{G}. \tag{5}$$

The process runs numerically till an arbitrary level set function $\tilde{G}$ becomes the signed distance function. In practice, $\tilde{G}$ estimates the solution to Eq. (1) at the given time. This estimate is not a signed distance function from the zero level set. Analytically, it is stated that in the limit $\tau \to \infty$, the solution of Eq. (5) tends to the unique viscosity solution of Eq. (4) without perturbation of the zero level set. However, practical computations have shown two difficulties concerning the reinitialization procedure given by Eq. (5).

(i) *Perturbation of the front*. In [12], it has been observed that after several iterations in discretized form of Eq. (5), the zero level set may move towards the nearest grid points which does not lie directly on the interface. The explanation of this effect [12] relies on the fact that in upwind methods, employed usually for integration of Eq. (5), the upwind differencing is performed according to the direction of the characteristics. This means that applying upwind differencing on grid points across the interface, the upwind property is violated, due to opposite directions of the characteristics propagation on the exterior and the interior. To overcome this drawback, the upwind fix across the zero level set was proposed in [12]; thereby the interface motion was effectively reduced. This approach was further developed in [13] for schemes of higher-order. In order to preserve area/volume during the level set reinitializations, a constraint was introduced in [14], in the form of the source term in Eq. (5). In [15,16], Eq. (5) was modified on the basis of the least-squares method allowing the displacement of the zero level set to be explicitly minimized within the reinitialization.

(ii) *The convergence problem* was addressed in [17]. For design of a stable scheme for (5), the sign function is usually smeared-out numerically. Such an approximation slows down the propagation speed of information from the zero level set. Since this information propagates along the normal direction, i.e. along the characteristics of the eikonal equation, the reduced speed of its propagation (less than one) requires, for the convergence, an increased number of iterations, much more than expected one for (5) with the unit propagation speed. Iterations become especially costly in the vicinity of the interface, where the smoothed sign function has intermediate values between -1 and 1.

In alternative to solving different forms of the reinitialization equation (5), there are other techniques, in which the reinitialization procedure is avoided, while the sign distance solution is preserved. As noted in [19] for example, one can obtain the signed distance as a solution to the evolutional equation if to correct properly the velocity field in Eq. (1), without changing the velocity on the interface. This idea was used in [20] for the formulation of the extension velocity method. In the extension velocity method, the main challenge is to introduce a new velocity field F^{ext} which coincides with the flow velocity field on the interface and is constant in the normal to the interface direction. Namely, F^{ext} is governed by the following boundary value problem:

$$\nabla F^{ext} \cdot \nabla G = 0 \quad \text{and} \quad F^{ext}|_{\sum} = F|_{\sum} = (\mathbf{u} \cdot \mathbf{n})|_{\sum}. \tag{6}$$

Then the G-equation written in terms of the normal velocity, i.e. Eq. (3), is solved with F^{ext}, instead of F. Similar approach was proposed in [21]. The difference between the approach from [21] and the extension velocity method [20] concerns the way of computing of F^{ext}. However, it has been mentioned in [22] that in complex flows, the computational cost of determining the extension velocity is high, and in some cases, the time-marching method for integration of Eq. (6) can lead to unexpected behavior. Further development of the extension velocity method was proposed and assessed in [22].

[1] Appendix A contains demonstration of how the reinitialization procedure works. A simple flow produced by one-dimensional strain is selected to illustrate clearly the key point of this procedure. Namely, it consists in the use of two G-fields at successive time steps: (i) the first field, with $|\nabla\tilde{G}| > 1$ is used to find the position of zero level set at current time; (ii) the second field, with $|\nabla G_{rein}| = 1$, is constructed from the knowledge of this position.