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Numerical integration techniques for discontinuous manufactured solutions

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ABSTRACT

When applying the method of manufactured solutions (MMS) on computational fluid dynamic software, determining the exact solutions and source terms for finite volume codes where the stored value is an integrated average over the control volume is non-trivial and not frequently discussed. MMS with discontinuities further complicates the problem of determining these values. In an effort to adapt the standard MMS procedure to solutions that contain discontinuities we show that Newton–Cotes and Gauss quadrature numerical integration methods exhibit high error, first order limitations. We propose a new method for determining the exact solutions and source terms on a uniform structured grid containing shock discontinuities by performing linearly and quadratically exact transformations on split cells. Transformations are performed on triangular and quadrilateral elements of a systematically divided discontinuous cell. Using a quadratic transformation in conjunction with a nine point Gauss quadrature method, a minimum of 4th order accuracy is achieved for fully general solutions and shock shapes. A linear approximation of curved shocks is also experimentally shown to be 2nd order accurate. The numerical integration method is then applied to a CFD code using simple discontinuous manufactured solutions which return consistent 1st order convergence values. The result is an important step towards being able to use MMS to verify solutions with discontinuities. This work also highlights the use of higher order numerical integration techniques for continuous and discontinuous solutions that are required for MMS on higher order finite volume codes.

1. Introduction

The method of manufactured solutions (MMS) has been used to verify computational fluid dynamic (CFD) codes using continuous, open flow examples since it was first developed by Steinberg and Roache in 1985 [1]. MMS rigorously verifies if a code is solving the governing equations correctly by checking the observed order of accuracy of the global solution [2]. To apply MMS, solutions are manufactured to the chosen governing equations (ex. Euler, Navier–Stokes equations). As it is highly unlikely that the manufactured solutions are an exact solution to the problem at hand, source terms are generated for

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the governing equations such that the manufactured solutions become exact. The numerical solution can then be compared to the manufactured solution to calculate the global discretization error and observed order of accuracy over multiple grid refinements.

An issue arises when performing MMS on finite volume codes since the stored value is an integrated average, as estimated by the fluxes around the control volume, and not the manufactured solution defined at a single point [3]. This is further complicated because manufactured solutions that are not analytically integrable are common. These solutions typically have many derivatives which helps ensure that the governing equations are fully exercised providing a more comprehensive test. Since source terms are derived from the exact solutions, allowing non-analytically integrable equations lifts a potentially difficult requirement. Trigonometric terms which are not analytically integrable and therefore do not have exact values, are frequently used to this end. Though usually not stated explicitly, current papers using MMS on finite volume codes, such as [2] and [4], likely use a midpoint approximation for determining the source terms and exact solutions. This 2nd order approximation of the integral is only appropriate for codes with order 2 or less. In Section 4 we show that it is simple to incorporate higher order methods that obtain a better first approximation and converge faster than the midpoint method. Using higher order approximations also extends the ability of MMS to higher order codes.

While MMS has been used primarily for simple continuous, open flow examples, using MMS with solutions containing discontinuities is still an open research issue [5]. Discontinuous MMS could be valuable for verifying code segments which may not activate for continuous flow solutions as well as to test the likely reduced order of a code when a shock appears; a potentially non-trivial case when dealing with linearly degenerate waves [6]. Another possible benefit is the testing of existing and new shock capturing schemes. Many of these schemes are tested using exact solutions to the Riemann problem consisting of simple waves separated by constant solutions and this form of testing may not be sufficient for obtaining the scheme's accuracy [7].

Determining the exact solutions and source terms for MMS with discontinuous solutions is a more difficult problem. It is shown in Section 3 that traditional Newton–Cotes methods have high error, first order limitations when cell faces are not shock aligned. Since discontinuous MMS has not yet been established in the literature, there are no guidelines for how a discontinuous problem should be defined. In [8] Roach suggests that grid convergence may be difficult to judge for a continuous solution with a sufficiently steep gradient. In this paper a piecewise solution technique is proposed to represent a shock and exactly integrable solutions are then used with varying shock shapes to test the accuracy of the integration approximations. In addition to CFD, this research could be of interest when dealing with any solution that contains a discontinuity. Discontinuities appear in many physical problems such as fluid–solid interfaces, dielectrics, and code to code couplings and are described generally in [9].

In Section 3 we prove in one dimensional space that the midpoint approximation typically used for MMS with finite volume codes results in a reduction of formal order when any discontinuity is present in the system. This same trend is also demonstrated experimentally in two dimensions for Simpson's and Gauss quadrature approximations. Due to both the low order restrictions and high uncertainty associated with using Newton–Cotes or Gauss quadrature methods in the presence of a discontinuity, a new method is explored in Section 4. Here we apply a cell transformation combined with a 6th order Gauss quadrature approximation. Each cell containing a discontinuity is systematically divided into triangular and quadrilateral elements and each element is then transformed using linearly and quadratically exact transformations. The 6th order Gauss quadrature method can then be used to obtain high order of accuracy in the estimation of both the exact solutions as well as the source terms.

In Section 5 the developed integration method is implemented into a CFD code. The code used is a cell centered, finite volume, 1st order, Eulerian scheme within the software AVUS (Air Vehicles Unstructured Solver). AVUS is used as a representative testing platform with the convenience of accessible source code. In CFD the manufactured solutions on either side of a discontinuity cannot be completely arbitrary and must be linked by physical equations. This is due to the nature of jump conditions which do not typically contain source terms as are normally used with MMS. Work completed by [10] and [11] have proposed solutions for dealing with source terms within the Riemann solver, but the code used in this paper contains no such modifications. We show that discontinuous manufactured solutions can be linked through Rankine–Hugoniot jump conditions. A simple discontinuous manufactured solution is then shown to converge with 1st order accuracy in the L_1 domain.

2. High order approximations

When performing MMS it is usually assumed that the analytical values to the exact solutions and source terms are known. This is not guaranteed to be true with finite volume codes, and approximations to the integral must be used. Despite this issue, MMS has been used to verify finite volume codes in the past and examples are given in [12] and [2]. Typically, the cell average is calculated using a midpoint approximation which is formally 2nd order accurate. While the midpoint approximation may be adequate for the code at hand, it should be noted that it is advantageous to institute a higher order scheme. Integration methods are frequently chosen based on where data already exists, but having the analytic functions means we are free to choose any method that best suits our needs. Two Newton–Cotes methods, the midpoint rule and Simpson's rule, are experimentally validated for a simulated manufactured solution, Eq. (1), in Table 1.

$$f(x, y) = e^y - \cos x \tag{1}$$