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A flux splitting method for the Euler equations

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ABSTRACT

The solution of the Euler and Navier–Stokes equations was the focus of this research. As with methods such as the AUSM-family, this work further attempted to construct an explicit upwind method that combines both simplicity and accuracy. In particular, with an appropriate choice of working variables, the aim was to develop a simple method that is based on the advection of the characteristic variables. The end result is a computationally efficient method valid for all homogeneous fluids with accuracy comparable to that of expensive flux-difference splitting methods. Good performance was extensively verified with Riemann test problems and results were on par with Roe's scheme. Furthermore, at least one formulation of the method passed all test problems satisfactory without requiring manual intervention. The method also illuminates some similarities between flux-difference splitting and the AUSM-family of methods.

1. Introduction

The accurate and efficient solution of the Euler and Navier–Stokes equations has been the focus of numerous researchers over the last few decades. Upwind schemes are well suited for the solution of hyperbolic conservation laws, and have been extensively used to solve the Euler equations [1,2]. These methods can broadly be categorised into flux vector splitting methods and Godunov-type methods.

In flux vector splitting methods the flux is usually split into positive and negative components, where these components usually adhere to specified spectral requirements. These methods are usually less expensive than Godunov-type methods. Early flux vector splitting methods such as the Steger–Warming and Van Leer splittings provided an efficient upwind discretisation. These simple splittings, however, lead to reduced accuracy [3,2,4]. Furthermore, splittings such as Van Leer's are not simultaneously diagonalisable [5]. Thus, numerical properties based on scalar advection equations such as stability criteria often cannot be used, especially for low speed flows. Extension of these splittings for general fluids is also not a straightforward task. Some of these shortcomings were addressed, see for example the variant of the Steger–Warming splitting found in [6]. It has the advantage of increased sonic point resolution as well as the exact solution of a one-dimensional stationary contact discontinuity.

On the other hand, Godunov-type methods such as Roe's approximate Riemann solver, which respects the full wave structure of the flux, are well known for their accuracy and have sharp stability estimates. Generally, this family of methods is usually complex and computationally expensive, and often requires an entropy fix for transonic expansion waves. Two-wave Riemann solvers based on the fastest and slowest waves such as the HLL and HLLE solvers provide improved efficiency

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[1,7]. In addition, these methods usually require no entropy fix [1]. However, intermediate waves are usually resolved poorly. For the one-dimensional Euler equations, this corresponds to excessive smearing of the contact discontinuity.

Hybrid methods have been introduced in an attempt to combine the favourable features of both types of methods. A popular idea has been to split the flux into advection and pressure components. Most notable of these, and of interest in this document, is the so-called AUSM (Advection Upstream Splitting Method) family of schemes. The original version was introduced by Liou and Steffen [4]. Following these ideas, numerous extended and modified versions have been developed [8–10], with the most recent version being AUSM$^+$-up [11]. Extension to two-phase flow can also be found in the literature (see for example [12]). These methods overcame many problems associated with traditional flux splitting methods, such as the resolution of stationary contact discontinuities and shocks, while maintaining simplicity. Recent versions, such as AUSM$^+$-up, often consist of various tunable parameters, which may be undesirable for general flow computation. Additionally, these methods do not always correspond to the wave structure of the characteristic variables. Accuracy and stability issues may thus arise. Based on Godunov-type discretisation, a framework for advection-pressure splittings was proposed [13]. The splitting of Zha–Bilgen (see [2] for details) was shown to be contained in that framework, while the authors were unable to do the same for the original AUSM method.

The purpose of this document is to present a finite volume method that is based on the full wave structure of the characteristic variables. Similar to the AUSM-family of schemes, the method was designed to split the flux into kinematic and acoustic parts. The method is valid for a general equation of state and maintains computational efficiency since it requires no evaluation and decomposition of a Jacobian matrix. The method, to some extent, puts the AUSM-family of schemes, in particular that of AUSM$^+$-up, into context with characteristic-based methods. The method was developed for the one-dimensional Euler equations, although extension to higher spatial dimensions is possible.

2. Preliminary theory

2.1. The Euler equations in one dimension

In one dimension, the Euler equations are given by

$$U_t + f(U)_x = 0, \quad U = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, \quad f = \begin{bmatrix} \rho u \\ mu + p \\ Eu + pu \end{bmatrix}. \tag{1}$$

t and x are the temporal and spatial coordinates respectively. ρ is the density, $m(=\rho u)$ the momentum per unit volume, E the total energy per unit volume, u the velocity and p the pressure. E can be written as $E = \rho(e + \frac{1}{2}u^2)$, where e is the internal energy per unit mass. The flux can be conveniently expressed as $f(U, p, u)$, i.e.

$$f = Uu + \begin{bmatrix} 0 \\ p \\ pu \end{bmatrix}, \tag{2}$$

where U appears explicitly in the formulation.

The Euler equations (1) can be written in non-conservative form

$$U_t + AU_x = 0, \tag{3}$$

where A is the Jacobian matrix of the flux F with respect to U. Furthermore, A can be diagonalised by $Q^{-1}AQ = \Lambda$ where Λ is a diagonal matrix whose elements λ_i are the eigenvalues of A. Q is the matrix whose columns r_i are right eigenvectors of A, Q^{-1} is the matrix whose rows l_i are left eigenvectors of A. With the change of variables $dW = Q^{-1}dU$, the Euler equations (1) can be written in characteristic form

$$W_t + \Lambda W_x = 0, \tag{4}$$

with $W = [\, w_1 \;\; w_2 \;\; w_3 \,]^T$ the characteristic variables and

$$\Lambda = \mathrm{diag}(u, u + a, u - a). \tag{5}$$

Here a is the local speed of sound. Hence w_1, w_2 and w_3 are advected respectively with speeds u, $u + a$ and $u - a$.

2.2. Finite volume discretisation

A finite volume discretisation of (1) is illustrated in Fig. 1 for the cell centred at x_i. This leads to the expression

$$\frac{dU_i}{dt} + \frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} = 0, \tag{6}$$

where the temporal discretisation is left unspecified. Δx is the spatial increment. $F_{i-1/2}$ and $F_{i+1/2}$ are the numerical fluxes at the faces located at $x_{i-1/2}$ and $x_{i+1/2}$ respectively. The primary task is to obtain suitable expressions for $F_{i-1/2}$ and $F_{i+1/2}$.