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Determining the effective resolution of advection schemes. Part I: Dispersion analysis

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ABSTRACT

The effective resolution of a numerical scheme describes the smallest spatial scale (largest wavenumber) that is completely resolved by that scheme. Using dispersion relation analysis allows the effective resolution of a numerical scheme for the advection equation to be calculated. The advection equation is a fundamental building block of dynamical cores of atmospheric and ocean models, and this analysis provides an indication of the effective resolution of the numerical methods used by dynamical cores. Using a variety of finite-difference schemes, the effect on effective resolution of using explicit diffusion and hyper-diffusion terms is examined. The choice of order-of-accuracy, and the time-stepping of the numerical scheme is also investigated with regard to effective resolution. Finally, we apply this analysis to methods that are commonly used in dynamical cores of atmospheric general circulation models, namely semi-Lagrangian and finite-volume methods.

1. Introduction

Advection schemes are an important building block of atmospheric dynamical cores. The dynamical core is the fluid dynamics component of an atmospheric model, and it solves the adiabatic governing equations (usually the primitive equations under certain approximations, for example hydrostatic balance) and the equations governing the transport of tracers. As well as solving the transport equations, advection schemes can be modified to solve conservation laws, such as the continuity equation for fluid density, or the vorticity equation. There are many different types of numerical methods that are used for advection in dynamical cores of general circulation models (GCMs), such as finite-difference [8], finite-volume [20,38], semi-Lagrangian [4,47], and spectral element [5]. It is important to understand the properties of different numerical methods, either to better understand the properties of existing advection schemes and dynamical cores, or to make an informed modeling choice when designing future models.

One property of a numerical method is the effective resolution. Whereas 'resolution' usually refers to the model's grid spacing, the effective resolution of a numerical scheme is generally defined as the smallest spatial scale (i.e. the largest wavenumber) that is 'fully resolved' by said numerical scheme. The shortest fully resolved wavelength, i.e. the effective resolution, is usually considerably larger than the grid spacing [41]. It is desirable to determine the effective resolution of a numerical scheme, and therefore the effective resolution of a model that makes use of the scheme. For example, in atmospheric modeling there is a desire to resolve features that are unresolved or only marginally resolved by current models,

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and thus improve weather forecasts and climate predictions [31]. As a higher effective resolution means that more features will be resolved by the model, increasing a model's effective resolution (through the choice of numerical methods) could prove a cheaper alternative than just doubling the grid resolution. This idea is closely related to the concept of 'equivalent resolution' as discussed in [46].

In addition, understanding the effects of explicit diffusion and filters on effective resolution provides insight into the tuning of diffusion coefficients (and the consequences of badly tuned parameters). With full GCMs the coupling of the subgrid-scale physical parameterization package and the "resolved" dynamical core is an important issue [6], and the physics parameterizations are often coupled to the dynamics at the grid scale. However, the dynamics do not truly resolve the grid scale, and it may be beneficial to add some of the physics to only the resolved scales i.e. the effective resolution [16]. Such GCM experiments with finer grid spacings in the dynamical core and coarser grids for the physics forcings were evaluated by Williamson [44]. For weather and climate models, composed of both dynamics and physics, Skamarock [31] suggested numerically calculating the effective resolution based upon the departure of the kinetic energy spectra from a given power law. However, analytical methods can be used to calculate the effective resolution of linear advection schemes, as proposed in this paper.

One tool to evaluate the properties of numerical schemes is dispersion relation analysis [28]. Linear dispersion relation analysis and von Neumann stability analysis of numerical schemes for atmospheric models has previously been performed for a variety of methods and equation sets [22,25,17,32]. This analysis can be used to investigate dispersive properties (such as group velocity and phase speed) and diffusive properties, and can be used to determine accuracy and stability of the numerical scheme [36,42]. Using this analysis to measure the effective resolution of a numerical method was introduced by Ullrich [40]. In [40], several different types of numerical methods (finite-volume, spectral element, spectral finite-volume, and discontinuous Galerkin) were analyzed for the linear wave equation with exact time integration, and their dispersive and diffusive properties were used to determine the effective resolution for different orders of accuracy. The aim of our paper is to modify this analysis for use with different time integration methods, to show the impact of the time integration scheme and the choice of timestep on the effective resolution of advection schemes. We also investigate the effect of explicit diffusion and hyper-diffusion terms on the diffusive and dispersive properties of a numerical scheme, and therefore the impact this diffusion has on effective resolution. We investigate these issues using simple finite-difference schemes, before applying the analysis to methods that are commonly used in transport schemes for dynamical cores; semi-Lagrangian and finite-volume methods. The assessment of the effective resolution of schemes for the advection equation is a first step towards investigating the effective resolution of the non-linear dynamics component of a GCM. There are many different types of advection schemes (see, for example, [30]). Our paper is not meant to be a comprehensive study of all advection schemes, but introduces the concept of calculating the effective resolution through a variety of possible choices in the algorithm. Although our focus is on numerical methods for atmospheric dynamical cores, the analysis can be applied to advection schemes that are used in any field of numerical analysis.

Some form of diffusion (either implicit in the numerics, as an explicitly added term, or in the form of a filter) is usually required for models solving non-linear governing equations on a fixed grid. In numerical studies of three-dimensional turbulence (large eddy simulation – LES) a subgrid model is required to dissipate kinetic energy, as this represents the effects of the unresolved flow on the resolved flow [23]. For two-dimensional flow it is the enstrophy which cascades downscale to unresolved scales, and therefore must be dissipated [13]. The atmosphere is strongly multiscale, with many interactions between these scales. Due to the effects of stratification and rotation, the atmosphere may resemble two-dimensional flow at large scales [1], before transitioning to three-dimensional flow at smaller scales. In dynamical cores of atmospheric models the diffusion is used to prevent the accumulation of potential enstrophy and kinetic energy at the grid scale, and also to dissipate tracer variance in the transport scheme [33,14]. This diffusion is often added in an ad-hoc way, and heavily tuned to provide optimal results [12]. For the constant velocity linear advection equation there are no diffusion effects in the true solution, although there are a number of numerical reasons that a modeler might chose to add diffusion to their scheme (for example to improve stability, to damp computational modes, or to ensure monotonicity). This means that although diffusion is undesired in the linear dispersion analysis for the linear advection equation, it is an essential part of the numerical methods that make up the dynamical cores. For this reason we consider the effects of some of the different forms of diffusion on the effective resolution of advection schemes.

This paper is structured as follows. Section 2 describes the one-dimensional linear advection equation and the dispersion relation and von Neumann analysis methodology. Using the dispersion relation analysis to determine the effective resolution of a number of numerical schemes is presented in Section 3, where we use finite-difference schemes to show the effects of order-of-accuracy, diffusion and time-stepping on effective resolution. We then turn our attention to numerical methods that are commonly used in dynamical cores, such as semi-Lagrangian and finite-volume methods. Conclusions are drawn in Section 4.

2. The advection equation

The one-dimensional advection equation is given as

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0, \tag{1}$$