Journal of Computational Physics 278 (2014) 497–508

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Determining the effective resolution of advection schemes. Part II: Numerical testing

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ARTICLE INFO

Article history:
Received 21 March 2014
Received in revised form 14 August 2014
Accepted 26 August 2014
Available online 3 September 2014

Keywords:
Effective resolution
Finite-difference methods
Test case
Dispersion analysis
Dynamical core

ABSTRACT

Numerical models of fluid flows calculate the resolved flow at a given grid resolution. The smallest wave resolved by the numerical scheme is deemed the effective resolution. Advection schemes are an important part of the numerical models used for computational fluid dynamics. For example, in atmospheric dynamical cores they control the transport of tracers. For linear schemes solving the advection equation, the effective resolution can be calculated analytically using dispersion analysis. Here, a numerical test is developed that can calculate the effective resolution of any scheme (linear or non-linear) for the advection equation.

The tests are focused on the use of non-linear limiters for advection schemes. It is found that the effective resolution of such non-linear schemes is very dependent on the number of time steps. Initially, schemes with limiters introduce large errors. Therefore, their effective resolution is poor over a small number of time steps. As the number of time steps increases the error of non-linear schemes grows at a smaller rate than that of the linear schemes which improves their effective resolution considerably. The tests highlight that a scheme that produces large errors over one time step might not produce a large accumulated error over a number of time steps. The results show that, in terms of effective-resolution, there is little benefit in using higher than third-order numerical accuracy with traditional limiters. The use of weighted essentially non-oscillatory (WENO) schemes, or relaxed and quasi-monotonic limiters, which allow smooth extrema, can eliminate this reduction in effective resolution and enable higher than third-order accuracy.

1. Introduction

Advection schemes perform an important role in the numerical models used for computational fluid dynamics. The advection equation describes passive transport, although many advection schemes can be used to solve conservation laws, such as the density or vorticity equations. They are a key component of a dynamical core, which solves the fluid dynamic equations in weather and climate general circulation models (GCMs). The advection equation is used to transport the many tracer species used in weather and climate studies and is strongly linked to the chemistry module and some subgrid-scale physical parameterizations [11,22].

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http://dx.doi.org/10.1016/j.jcp.2014.08.045
0021-9991/

It is well known that the smallest resolved waves of a numerical scheme for the advection equation are often significantly larger than the grid spacing [34]. For weather and climate models this means that many atmospheric features, that are of the order of the grid scale, are not resolved by the model. Determining the smallest resolved wave of an atmospheric model, which we define as the effective resolution, provides insight into which scales are believable [12]. This can be used to determine the grid spacing required to properly represent atmospheric features. Knowledge of the effective resolution of a numerical scheme also informs the coupling of a dynamical core to subgrid scale physical parameterizations, which provide a forcing mechanism at the grid scale. Increasing the effective resolution of a model by using higher-order numerical methods might prove beneficial in terms of cost rather than just increasing the grid resolution (similar to the idea of equivalent resolution [36]).

Part I of our series of papers [8] used dispersion relation analysis to calculate the effective resolution of a number of schemes for the linear advection equation. Dispersion relation and von Neumann analysis are tools that have been used to analyze many numerical methods [14,20,21,24,35]. If a scheme's dispersion properties match those of the governing equation at a given wave number, and within a given error tolerance, then that wave number is classified as resolved [8,32].

One drawback of the dispersion analysis is that it can only be applied to linear schemes. For the advection equation there are many different types of numerical schemes (see, for example, [13,23,25]). Many advection schemes contain non-linear components, such as limiters or filling algorithms, and as such the effective resolution of these schemes cannot be assessed by dispersion analysis. Here, we present a numerical test that can be used by both linear and non-linear advection schemes to calculate their effective resolution. The numerical test analyses the method over a number of time steps, which will have an impact on the non-linear schemes, as numerical schemes that perform poorly over a single time step might not produce a large accumulated error over a number of time steps. We use this method to investigate the effect that non-linear components, such as limiters, have on the effective resolution of advection schemes.

The analysis and numerical testing in our paper focuses on the linear advection equation, allowing easy comparison with the dispersion analysis performed by [8]. The advection equation is reviewed in Section 2, along with a recap of the analysis of [8]. In Section 3 we develop idealized numerical tests to allow the calculation of the effective resolution of any advection scheme. Section 4 shows the results from the numerical testing of limited schemes, while Section 5 provides the summary and conclusions.

2. The advection equation and dispersion analysis

The one-dimensional advection equation is given as

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0, \tag{1}$$

where $q(x,t)$ is a tracer mixing ratio, u is the velocity (in this paper we choose u constant, with $u = 1$), x is the spatial direction and t is time. Note that all quantities are dimensionless in this paper. The solution to the constant velocity advection equation is known, and is given as

$$q_T(x,t) = q_0(x - ut), \tag{2}$$

where q_0 is the initial tracer and the subscript T indicates the true solution.

The effective resolution describes the smallest wave (largest wave number) that is fully resolved by a numerical scheme. To calculate the effective resolution using dispersion analysis we follow [8]. For the one-dimensional advection equation with constant velocity the true amplitude factor, $|\Gamma|$, and dispersion relation are given as

$$|\Gamma| = 1, \qquad \omega = uk, \tag{3}$$

where k is the spatial wave number, and ω is the frequency. To calculate the effective resolution, the scheme's amplitude factor ($|\Gamma_N|$) and dispersion relation (ω_N) are compared with the true amplitude factor and dispersion relation for all wave numbers. The amplitude factor and dispersion relation are calculated by substituting the wavelike solution

$$q_j^n = \hat{q}\exp\left(i(kx_j - \omega t_n)\right) \tag{4}$$

into the discretization. Here n and j are the temporal and spatial indices, i is the imaginary unit and $\hat{q}$ is the amplitude. The amplitude factor is calculated as $|\Gamma| = |\exp(-i\omega\Delta t)|$, for a time step Δt. Wave number k is defined as fully resolved if

$$\frac{||\Gamma| - |\Gamma_N||}{|\Gamma|} \le \epsilon, \qquad \frac{|\omega - Re(\omega_N)|}{|\omega|} \le \epsilon, \tag{5}$$

for all wave numbers $\le k$ at some error threshold ϵ. Following [8,32], we use $\epsilon = 0.01$, i.e. a scheme must be within 99% of the true amplitude factor and dispersion relation. We are interested in the effective resolution of a scheme as it transports a quantity over the distance of one grid box, Δx. To do this the amplitude factor is taken to the power m, where $m = 1/c$ for Courant number $c = u\Delta t/\Delta x$ (i.e. m is the number of time steps required to transport a quantity one full grid box).